

THE HIROTA τ -FUNCTION AND WELL-POSEDNESS OF THE KDV EQUATION WITH AN ARBITRARY STEP LIKE INITIAL PROFILE DECAYING ON THE RIGHT HALF LINE

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ABSTRACT. We are concerned with the Cauchy problem for the KdV equation on the whole line with an initial profile V_0 which is decaying sufficiently fast at $+\infty$ and arbitrarily enough (i.e., no decay or pattern of behavior) at $-\infty$. We show that this system is completely integrable in a very strong sense. Namely, the solution $V(x, t)$ admits the Hirota τ -function representation

$$V(x, t) = -2\partial_x^2 \log \det(I + \mathbb{M}_{x,t}) \quad (0.1)$$

where $\mathbb{M}_{x,t}$ is a Hankel integral operator constructed from certain scattering and spectral data suitably defined in terms of the Titchmarsh-Weyl m -functions associated with the two half-line Schrödinger operators corresponding to V_0 . We show that $V(x, t)$ is real meromorphic with respect to x for any $t > 0$. We also show that under a very mild additional condition on V_0 representation (0.1) implies a strong well-posedness of the KdV equation with such V_0 's. Among others, our approach yields some relevant results due to Cohen, Kappeler, Khruslov, Kotlyarov, Venakides, Zhang and others.

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1. INTRODUCTION

Soliton theory, a major achievement of 20th century mathematics, originated in 1965 from the fundamental Gardner-Greene-Kruskal-Miura discovery of what we now call the inverse scattering transform (IST) for the Korteweg-de Vries (KdV) equation. Conceptually, the IST is similar to the Fourier transform. In the context of the Cauchy problem for the KdV equation on the full line

$$\partial_t V - 6V\partial_x V + \partial_x^3 V = 0 \quad (1.1)$$

$$V(x, 0) = V_0(x) \quad (1.2)$$

the IST method consists, as the standard Fourier transform method, of the following three steps:

- (1) the direct transform mapping the initial data $V_0(x)$ to a new set of variables S_0 in which (1.1) turns into a very simple first order linear ordinary equation for $S(t)$ with the initial condition $S(0) = S_0$;
- (2) solve then this linear ordinary differential equation for $S(t)$;
- (3) apply the inverse transform to find $V(x, t)$ from $S(t)$.

In its original edition due to Gardner-Greene-Kruskal-Miura, S_0 was the set of scattering data associated with the pair of Schrödinger operators (H, H_0) where $H_0 = -\partial_x^2$ and $H = H_0 + V_0$. Moreover, this procedure comes with a beautiful formula

$$V(x, t) = -2\partial_x^2 \log \det(I + \mathbb{M}_{x,t}), \quad (1.3)$$

where $\mathbb{M}_{x,t}$ is a two parametric family of integral operators explicitly constructed in terms of $S(t)$.

Similar methods have also been developed for many other evolution nonlinear partial differential equations (PDE), which are referred to as completely integrable.¹

Strictly speaking, a complete integrability of (1.1)-(1.2) was originally established for V_0 's rapidly decaying (short range) at $\pm\infty$ (i.e. in the case of the scattering theoretical situation for (H, H_0)). A suitable analog of IST for the physically important case of periodic V_0 's was found around 1974 by Novikov [40]. It is based upon the Floquet theory for the periodic (Hill) Schrödinger operator and looks very different from the case of decaying initial data. There is a conjecture

¹There is no precise meaning of "complete integrability" but the question "What is integrability?" has drawn much attention (see e.g. the charming survey [26] by Its).

however that these two cases can actually be unified [1]. Some deep related results are given in [54] and [12].

Thus, the KdV equation (as well as other completely integrable by the IST PDEs) are completely integrable essentially only in these two cases. In fact, the fundamental question about whether any (physically significant) well-posed problem (initial value, boundary value, etc.) for equation (1.1) can be solved by a suitable IST, has been raised in one form or another by many (see e.g. [39], [1], [29]). A large amount of effort has been put into developing the IST for (1.1) on an interval (finite and semi-infinite). There seems to be no consensus on whether (1.1) is indeed completely integrable in this case but some relevant deep results have been obtained (see, e.g., [16] and the literature therein).

The specific concern of this paper is the problem (1.1)-(1.2) with V_0 outside of classes of rapidly decaying or periodic functions. Many of known related results are devoted to some sort of “hybrids” of the two well-developed cases. Namely, a physically important case of an initial profile V_0 in (1.2) which is a short-range perturbation of a step function (a bore type initial profile) appears to have received the most attention (see e.g. [25], [7], [53], [35] to name just a few and the extensive literature cited therein). Another important case where IST works is (1.1)-(1.2) with V_0 representable as a short range perturbation of a half-periodic² potential (see e.g. [35]). A rigorous comprehensive treatment of steplike short-range perturbations of finite-gap solutions was recently given in [14]. We also refer to [14] for a large account of literature on initial profiles for which IST works.

Certain cases of slowly decaying profiles have also received considerable attention (see e.g. [33], [34], [18], [38]), but this situation is far from being well-understood.

More literature and discussion of the results obtained therein will be offered in the main body of the paper. We only mention that in all the situations above the spectrum of the underlying Schrödinger operator $-\partial_x^2 + V_0$ is much more complicated resulting in new phenomena (infinite train of solitons, solitons on periodic backgrounds, singular solutions, etc.). Consequently, much more complicated tools and harder analysis are required.

We mention that the question of well-posedness³ of (1.1)-(1.2) is a serious issue (see, e.g. [50]). In the literature on IST this issue is frequently avoided.

The main concern of the present paper is to show that (1.1)-(1.2) is completely integrable in a very strong sense for V_0 's rapidly decaying at $+\infty$ and essentially arbitrary at $-\infty$. We will also call such V_0 's steplike initial profiles. Under very mild conditions on initial data V_0 (expressed in terms of the spectrum of the underlying Schrödinger operator $-\partial_x^2 + V_0(x)$), we prove that the representation (1.3) can be extended to our case. We also establish the analytic smoothing effect and well-posedness of (1.1)-(1.2) for initial profiles under consideration. Emphasize that, as opposed to the relevant previous works, we treat initial profiles which need not have specific asymptotic behavior at $-\infty$. Our approach is based upon IST techniques and suitable limiting argument. Note that the limiting procedures which we employ allow us to recycle many results from the classical IST method and, in fact, bypass analysis of the relevant Fredholm integral equation or Riemann-Hilbert problem and work directly with a Hankel integral operator.

²I.e. a function which is periodic on a half-line and zero on the other.

³I.e. existence, uniqueness and continuous dependence on the initial data.

The paper is organized as follows. In Section 2 we merely list some of our notation and conventions. In Section 3, we review some basics of the full line and half line Schrodinger operators and the full line short range scattering theory. In Section 4 we put together some well-known facts on the KdV equation pertinent to the present paper and specifies what we mean by the solution to the Cauchy problem for the KdV equation. Section 5 is devoted to defining a right reflection coefficient from the right incident for arbitrary potentials. In Section 6 we specify in what sense we understand Fredholm determinants (Definition 6.1) and prove two important lemmas. In Section 7 we introduce and study a Hankel integral operator particularly important in the context of the IST. The main results (Theorems 8.1 and 8.6) are given in Section 8 and Section 9 is devoted to related discussions and corollaries. We also state some open problems.

2. NOTATION AND PRELIMINARIES

We adhere to standard terminology accepted in Analysis. Namely, $\mathbb{R}_\pm := [0, \pm\infty)$, \mathbb{C} is the complex plane,

$$\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}.$$

Unless otherwise stated, we use the subscript \pm to indicate objects somehow related to \mathbb{R}_\pm or \mathbb{C}_\pm . E.g. m_- (m_+) denotes the Titchmarsh-Weyl m -function (introduced below) associated with \mathbb{R}_- (\mathbb{R}_+). The bar \bar{z} denotes the complex conjugate of z .

We use $\|\cdot\|_X$ to denote the norm in a Banach (Hilbert) space X . We extensively use Lebesgue spaces ($1 \leq p < \infty$, $d\mu$ is a non-negative measure on a set S)

$$\begin{aligned} L^p(S, d\mu) &:= \left\{ f : \|f\|_{L^p(S)} := \left(\int_S |f(x)|^p d\mu(x) \right)^{1/p} < \infty \right\}, \\ L^\infty(S) &:= \left\{ f : \|f\|_{L^\infty(S)} := \operatorname{ess\,sup}_{x \in S} |f(x)| < \infty \right\}, \\ L^p_{\operatorname{loc}}(S) &:= \{\cap L^p(\Delta) : \Delta \subset S\}. \end{aligned}$$

and abbreviate them in particular cases as follows (S will typically be \mathbb{R} or \mathbb{R}_\pm):

$$\begin{aligned} L^p(S, dx) &=: L^p(S) \\ L^p(\mathbb{R}_\pm, d\mu) &=: L^p_\pm(d\mu), \quad L^p(\mathbb{R}, d\mu) =: L^p(d\mu) \\ L^p(\mathbb{R}_\pm) &=: L^p_\pm, \quad L^p(\mathbb{R}) =: L^p \end{aligned}$$

Next, \mathfrak{S}_2 denotes the Hilbert-Schmidt class of linear operators A :

$$\mathfrak{S}_2 = \left\{ A : \|A\|_{\mathfrak{S}_2} := \operatorname{tr}(A^* A) < \infty \right\}$$

and \mathfrak{S}_1 is the trace class:

$$\mathfrak{S}_1 = \left\{ A : \|A\|_{\mathfrak{S}_1} := \operatorname{tr}(A^* A)^{1/2} < \infty \right\}.$$

$\operatorname{Spec}(A)$ stands for the spectrum of an operator A and $\operatorname{Spec}_{\operatorname{ac}}(A)$, $\operatorname{Spec}_d(A)$ denote the absolutely continuous (a.c.), discrete components of the (self-adjoint) operator A .

Throughout the paper

$$\begin{aligned} (\mathcal{F}f)(\lambda) &= \mathcal{F}f(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda x} f(x) dx \\ (\mathcal{F}^{-1}f)(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda x} f(x) dx \end{aligned}$$

stand for the Fourier and the inverse Fourier transforms of a tempered distribution f respectively. If $f \in L^2$ and $\text{Supp } f \subseteq \mathbb{R}_{\pm}$ then $(\mathcal{F}f)(\lambda) \in H_{\pm}^2$. Recall that $H_{\pm}^p, p > 0$, stands for the Hardy class of analytic on \mathbb{C}_{\pm} functions f such that

$$\sup_{\pm y > 0} \int_{\mathbb{R}} |f(x + iy)|^p dx < \infty.$$

Each H^p -function f admits the estimate (see, e.g. [17])

$$|f(\lambda)| \lesssim \frac{\|f\|_{H_{\pm}^p}}{\text{Im } \lambda} \quad (2.1)$$

where we have used a convenient convention to write

$$x \lesssim y \iff x \leq Cy$$

with some $C > 0$ independent of x and y .

Some other miscellaneous notation: $\chi_S(x)$ is the characteristic function of a set S , i.e.

$$\chi_S(x) := \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}.$$

In particular $\chi_{\pm} := \chi_{\mathbb{R}_{\pm}}$ is the Heaviside function of \mathbb{R}_{\pm} .

The following short hand notation will help us keep bulky formulas under control

$$\begin{aligned} \int_{\mathbb{R}} &=: \int \\ (fg)(x) &:= f(x)g(x) \end{aligned}$$

and

$$\langle x \rangle := \sqrt{1 + |x|^2}$$

for an inhomogeneous distance.

3. THE 1D SCHRÖDINGER OPERATOR, SCATTERING AND ALL THAT

As its title suggests, in this section we review some basics of the full line and half line Schrodinger operators and the full line short range scattering theory.

3.1. Schrödinger operators on the line. As well known, the inverse scattering transform (IST) method for the KdV equation is based upon the direct and inverse scattering for the one-dimensional Schrödinger operator. In this section we briefly introduce the 1D Schrödinger operator referring the reader to [52] for precise statements. Through the paper

$$H_0 = -\partial_x^2 \quad (3.1)$$

is the free (unperturbed) Schrödinger operator on the Hilbert space L^2 . Given a real locally integrable function V , called a potential, define on L^2 the (perturbed) Schrödinger operator:

$$H = H_0 + V(x) = -\partial_x^2 + V(x) \quad (3.2)$$

and two half-line Schrödinger operators with the Dirichlet boundary condition at $x = 0$

$$H_{\pm}^D = -\partial_x^2 + V(x) \text{ on } L_{\pm}^2 \text{ with } u(\pm 0) = 0. \quad (3.3)$$

We shall assume that each Schrödinger operator considered in this paper is self-adjoint on its natural domain in L^2 . This assumption automatically puts a certain restriction on V 's:

Hypothesis 3.1. (1) *Reality*

$$\overline{V}(x) = V(x)$$

(2) *Local integrability*

$$V \in L_{loc}^1$$

(3) *V is limit point case at $\pm\infty$*

$$V \in l.p.(\pm\infty)$$

Hypothesis 3.1 means that the minimal operator generated by the differential expression $-\partial_x^2 + V(x)$ in the space L_{\pm}^2 has deficiency indices $(1, 1)$ as opposed to the limit circle case when the deficiency indices are $(2, 2)$. While no explicit description (e.g. in terms of potentials) of the limit point/circle classification is currently available but it is well-known that the class $V \in l.p.(\pm\infty)$ is extremely broad. In particular, all physically meaningful initial profiles (no decay of any kind is assumed) in the KdV equation are limit point at $\pm\infty$. We will refer to potentials subject to Hypothesis 3.1 as arbitrary.

3.2. The Titchmarsh-Weyl m -function. The material of this subsection is classical and standard (see, e.g. [52]). Assuming Hypothesis 3.1, consider

$$-\partial_x^2 u + V(x)u = zu, x \in \mathbb{R}_{\pm}.$$

If $V \in l.p.(\pm\infty)$ then there exists a unique (up to a multiplicative constant) solution, called Weyl, such that $\Psi_{\pm}(x, z) \in L_{\pm}^2$ for each $z \in \mathbb{C}_+$.

Definition 3.2. *The function*

$$m_{\pm}(z) = \frac{\partial_x \Psi_{\pm}(0, z)}{\Psi_{\pm}(0, z)}, \quad z \in \mathbb{C}_+$$

is called (Dirichlet, principal) Titchmarsh-Weyl m -function.

Alternatively, the m -function can be defined

$$m_{\pm}(z) = \lim_{0 < x < y \rightarrow 0} \partial_{xy}^2 G_{\pm}(x, y; z)$$

with $G_{\pm}(x, y; z)$ the Green's function of the corresponding half line Schrödinger operator. Properties of the m -function include

(1) The Herglotz property: $m : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ and analytic

- (2) The Herglotz representation theorem: there is a non-negative measure μ subject to $\int \langle \lambda \rangle^{-2} d\mu(\lambda)$ such that

$$m(z) = \operatorname{Re} m(i) + \int_{\mathbb{R}} \frac{t+z}{t-z} \frac{d\mu}{1+t^2}. \quad (3.4)$$

The measure μ is the spectral measure of H_{\pm}^D introduced by (3.3) and can be computed by the Herglotz inversion formula⁴

$$d\mu = \frac{1}{\pi} \operatorname{Im} m(t+i0) dt.$$

- (3) The Borg-Marchenko Uniqueness Theorem:

$$m_1 = m_2 \implies V_1 = V_2.$$

3.3. Scattering theory for the Schrödinger operator on the line. To fix our notation and terminology we give a brief introduction to 1D scattering theory for the Schrödinger operator. We refer the reader to the classical paper [11] (where the notation slightly differs from ours though). Through this section we deal with a pair (H, H_0) of Schrödinger operators (3.2) and (3.1) with a Faddeev potential V . I.e.

$$V \in L^1(\langle x \rangle dx) = \left\{ f : \int (1+|x|) |f(x)| dx < \infty \right\}.$$

Under this assumption on V one has a typical scattering theoretical situation which means that all four wave operators and the scattering operator for the pair (H, H_0) exist. In particular, the absolutely continuous (a.c.) part of H is unitary equivalent to H_0 . For $\operatorname{Spec}(H)$ we have:

$$\operatorname{Spec}(H) = \operatorname{Spec}_d(H) \cup \operatorname{Spec}_{ac}(H),$$

where the discrete spectrum $\operatorname{Spec}_d(H) = \{-\kappa_n^2\}_{n=1}^N$ is negative, simple and

$$N \leq 1 + \|V\|_{L^1(\langle x \rangle dx)}$$

and the absolutely continuous (a.c.) spectrum $\operatorname{Spec}_{ac}(H) = \mathbb{R}_+$ and of multiplicity two (with no embedded eigenvalues).

Since the a.c. spectrum of H is of uniform multiplicity two, the scattering matrix S (the scattering operator in the spectral representation of H_0) is a two by two unitary matrix

$$S(\lambda) = \begin{pmatrix} T(\lambda) & R(\lambda) \\ L(\lambda) & T(\lambda) \end{pmatrix}, \lambda^2 \in \operatorname{Spec}_{ac}(H) = \mathbb{R}_+,$$

where T, L and R denote respectively the transmission, reflection coefficients from the left and right incident. Due to unitarity of S one has (for a.e. $\lambda \in \mathbb{R}$)

$$|T(\lambda)|^2 + |R(\lambda)|^2 = 1, |T(\lambda)|^2 + |L(\lambda)|^2 = 1 \quad (3.5)$$

$$\overline{T(\lambda)} R(\lambda) + T(\lambda) \overline{L(\lambda)} = 0 \quad (3.6)$$

$$T(-\lambda) = \overline{T(\lambda)}, R(-\lambda) = \overline{R(\lambda)}, L(-\lambda) = \overline{L(\lambda)}.$$

⁴Through the paper we use the convention

$$\operatorname{Im} f(t+i0) dt := w - \lim_{\varepsilon \rightarrow +0} \operatorname{Im} f(t+i\varepsilon) dt.$$

The quantities T, L and R are related to the existence of special solutions ψ_{\pm} (called Jost or Faddeev) to the (stationary) Schrödinger equation

$$-\partial_x^2 u + V(x)u = \lambda^2 u, \lambda \in \mathbb{R},$$

asymptotically behaving as

$$\begin{aligned} \psi_+(x, \lambda) &\sim \begin{cases} T(\lambda) e^{i\lambda x}, & x \rightarrow \infty, \\ e^{i\lambda x} + L(\lambda) e^{-i\lambda x}, & x \rightarrow -\infty, \end{cases} \\ \psi_-(x, \lambda) &\sim \begin{cases} e^{-i\lambda x} + R(\lambda) e^{i\lambda x}, & x \rightarrow \infty, \\ T(\lambda) e^{-i\lambda x}, & x \rightarrow -\infty. \end{cases} \end{aligned} \quad (3.7)$$

Scattering solutions $\psi_{\pm}(x, \lambda)$ can be obtained by solving the Volterra integral equations⁵

$$y_{\pm}(x, \lambda) = 1 \pm \int_x^{\pm\infty} \frac{e^{2i\lambda|s-x|} - 1}{2i\lambda} V(s) y_{\pm}(s, \lambda) ds \quad (3.8)$$

for

$$y_{\pm}(x, \lambda) := \frac{\psi_{\pm}(x, \lambda)}{T(\lambda)} e^{\mp i\lambda x}.$$

For the transition coefficients T, R, L one has

$$T(\lambda) = \left(1 - \frac{1}{2i\lambda} \int V(x) y_+(x, \lambda) dx\right)^{-1} \quad (3.9)$$

$$R(\lambda) = \frac{T(\lambda)}{2i\lambda} \int e^{-2i\lambda x} V(x) y_-(x, \lambda) dx \quad (3.10)$$

$$L(\lambda) = \frac{T(\lambda)}{2i\lambda} \int e^{2i\lambda x} V(x) y_+(x, \lambda) dx. \quad (3.11)$$

Remark 3.3. The assumption $V \in L^1(\langle x \rangle dx)$ can be relaxed to $V \in L^1$ in most of statements of this section. The number of negative eigenvalues (bound states) $\{-\kappa_n^2\}_{n \geq 1}$ may become infinite accumulating to 0. The latter implies more complicated behavior of the scattering matrix at $\lambda = 0$.

The following facts from [11] will be important:

- (1) If $V \in L^1(\langle x \rangle^2 dx)$ and generic, i.e.

$$T(\lambda) = \alpha\lambda + o(\lambda), \lambda \rightarrow 0, \alpha \neq 0, \quad (3.12)$$

then the function

$$\begin{aligned} f(\lambda) &:= \sqrt{2\pi} \frac{T(\lambda/2)}{i\lambda} = \sqrt{2\pi} \left(i\lambda - \int V(x) y_+(x, \lambda/2) dx \right)^{-1} \\ &= O(1/\lambda), \lambda \rightarrow \infty, \end{aligned} \quad (3.13)$$

and is continuous on \mathbb{R} . A potential $V \in L^1(\langle x \rangle^2 dx)$ for which (3.12) doesn't hold is called exceptional and it can be turned into a generic one by an arbitrary small deformation.

⁵In the literature, y_{\pm} are typically denoted by $m_{1,2}$ and called Faddeev or Jost functions. In our exposition the letter m is reserved for the m -function.

(2) There exists a function g subject to

$$|g(x)| \leq |V(x)| + \text{const } W(x), \quad W(x) := \begin{cases} \int_x^\infty |V|, & x \geq 0 \\ \int_{-\infty}^x |V|, & x < 0 \end{cases} \quad (3.14)$$

such that

$$R(\lambda/2) = f(\lambda) \left(\mathcal{F}^{-1} g \right) (\lambda) \quad (3.15)$$

where f is defined by (3.13).

(3) If $V \in L^1(\langle x \rangle^2 dx)$ then

$$\|\partial_\lambda y_+(x, \cdot)\|_{L^\infty} \lesssim \langle x \rangle^2. \quad (3.16)$$

We will need the following technical lemma.

Lemma 3.4. *If $V \in L^2(\langle x \rangle^2 dx)$ and generic then $\partial_\lambda R \in L^2$.*

Proof. From (3.15)

$$\partial_\lambda R(\lambda/2) = \partial_\lambda f(\lambda) \left(\mathcal{F}^{-1} g \right) (\lambda) + f(\lambda) \partial_\lambda \left(\mathcal{F}^{-1} g \right) (\lambda)$$

and hence

$$\begin{aligned} \|\partial_\lambda R\|_{L^2} &\lesssim \|\partial_\lambda f\|_{L^\infty} \left\| \mathcal{F}^{-1} g \right\|_{L^2} + \|f\|_{L^\infty} \left\| \mathcal{F}^{-1} xg \right\|_{L^2} \\ &= \|\partial_\lambda f\|_{L^\infty} \|g\|_{L^2} + \|f\|_{L^\infty} \|xg\|_{L^2}. \end{aligned}$$

Due to (3.13) and (3.14) $\|f\|_{L^\infty}$ and $\|g\|_{L^2}$ are both finite. The condition $V \in L^2(\langle x \rangle^2 dx)$ implies by (3.14) that $\|xg\|_{L^2} \lesssim \|g\|_{L^2(\langle x \rangle^2 dx)}$ is finite. Indeed, setting $W_+(x) := \chi_+(x) \int_x^\infty |V|$ and $W_-(x) := \chi_-(x) \int_{-\infty}^x |V|$ one has

$$\begin{aligned} \|W_\pm\|_{L_\pm^2}^2 &= \int_0^\infty \left(\int_x^\infty |V(\pm s)| ds \right)^2 dx = 2 \int_0^\infty x |V(\pm x)| \int_x^\infty |V(\pm s)| ds dx \\ &\leq 2 \|xV\|_{L_\pm^2} \|W_\pm\|_{L_\pm^2} \end{aligned}$$

and hence

$$\|W_\pm\|_{L_\pm^2} \leq 2 \|xV\|_{L_\pm^2} \leq 2 \|V\|_{L_\pm^2(\langle x \rangle^2 dx)}.$$

We therefore have $g \in L^2(\langle x \rangle^2 dx)$ and hence $\|xg\|_{L^2}$ is finite. It only remains to show that $\partial_\lambda f$ is bounded. By direct differentiation of (3.13) one has

$$\partial_\lambda f(\lambda) = \frac{-1}{\sqrt{2\pi}} f^2(\lambda) \left(i - \int V(x) \partial_\lambda y_+(x, \lambda/2) dx \right)$$

and by (3.16)

$$|\partial_\lambda f| \lesssim |f|^2 \left(1 + C \|V\|_{L^1(\langle x \rangle^2 dx)} \right)$$

and due to (3.13) the statement is proven. \square

4. THE KDV, IST AND ALL THAT

In this section we merely put together some well-known information on the KdV equation pertinent to the present paper. Definition 4.2 below specifies what we mean by the solution to the Cauchy problem for the KdV equation.

4.1. The classical Inverse Scattering Transform (IST). To specify our notation and state pivotal equations we briefly outline basic ideas of the IST methods originated from the seminal 1965 work of Gardner-Greene-Kruskal-Miura (see, e.g. [1] and the very extensive literature cited therein). In the context of the initial value problem for the KdV equation on \mathbb{R}

$$\partial_t V - 6V\partial_x V + \partial_x^3 V = 0 \quad (4.1)$$

$$V(x, 0) = V_0(x) \quad (4.2)$$

where $V(x, t)$ is subject to⁶ ($l = 0, 1, 2, 3$)

$$\sup_{t \geq 0} \left\| \partial_x^l V(x, t) \right\|_{L^1(\langle x \rangle dx)} < \infty$$

the classical inverse scattering formalism for (4.1) – (4.2) goes as follows. Associate with (4.1) a one parametric family of full line Schrödinger operators $H(t) = -\partial_x^2 + V(x, t)$. Form now the scattering data

$$\mathcal{S}(t) := \{R(\lambda, t), \{-\kappa_n^2(t), c_n(t)\}\} \quad (4.3)$$

where $\{c_n(t)\}$ are the norming constants corresponding to bound states $\{-\kappa_n^2(t)\}$. It is well-known that the map $V(x, t) \rightarrow \mathcal{S}(t)$ is one-to-one.

The fundamental fact of inverse scattering formalism is that the map (time evolution) $t \rightarrow \mathcal{S}(t)$ has a very simple form

$$R(\lambda; t) = R(\lambda)e^{8ik^3t}, \kappa_n(t) = \kappa_n, c_n(t) = c_ne^{-4\kappa_n^3t}. \quad (4.4)$$

The problem (4.1) – (4.2) can now be solved in three steps. Solve the direct scattering problem $V_0(x) \rightarrow \mathcal{S}(0)$. Find next the time evolution $\mathcal{S}(t)$ by (4.3) and (4.4) and finally solve the inverse scattering problem $\mathcal{S}(t) \rightarrow V(x, t)$. The last step can be done by any applicable method. For instance, one can solve $\mathcal{S}(t) \rightarrow V(x, t)$ as a Riemann-Hilbert problem

$$\psi_+(x, -\lambda, t) + R(\lambda)e^{8ik^3t}\psi_+(x, \lambda, t) = T(\lambda)\psi_-(x, \lambda, t), \lambda \in \mathbb{R}$$

for ψ_{\pm} . The function

$$V(x, t) = \frac{\partial_x^2 \psi_{\pm}(x, \lambda, t)}{\psi_{\pm}(x, \lambda, t)} + \lambda^2$$

then solves (4.1), the procedure being independent of the choice of \pm . Alternatively, one can solve the inverse problem $\mathcal{S}(t) \rightarrow V(x, t)$ by means of the Marchenko procedure⁷ which essentially boils down to the nice formula

$$V(x, t) = -2\partial_x^2 \log \det(I + \mathbb{M}_{x,t})$$

where $\mathbb{M}_{x,t} : L_+^2 \rightarrow L_+^2$ is a two parametric family of integral operators

$$(\mathbb{M}_{x,t}f)(y) = \int_0^\infty M_{x,t}(y+s)f(s)ds, \quad f \in L_+^2, \quad (4.5)$$

⁶such solutions are referred to as rapidly decaying. For simplicity we call them short range.

⁷This procedure is also referred to as the Gelfand-Levitan-Marchenko.

with the kernel

$$M_{x,t}(\cdot) := M(\cdot + 2x, t), \quad (4.6)$$

$$M(y, t) := \sum_{n=1}^N c_n^2 e^{8\kappa_n^3 t} e^{-\kappa_n y} + \frac{1}{2\pi} \int e^{i\lambda y} R(\lambda) e^{8i\lambda^3 t} d\lambda.$$

Definition 4.1. *The operator $\mathbf{M}_{x,t}$ defined by (4.5), (4.6) is called a (time evolved) Marchenko operator⁸ associated with the scattering data (4.3) and (4.4).*

We have actually considered the left Marchenko operator. The right Marchenko operator can be introduced in a similar manner but will not admit a proper generalization to our setting.

Over the last forty years soliton theory has experienced a rapid development through efforts by the math, science and engineering communities and the literature on the subject is enormously extensive and diverse. Some literature relevant to our consideration have already been given in Introduction and some more will be given below. In addition to this we mention here that certain IST type schemes are also available for the so-called finite gap algebro-geometric solutions to KdV. (see e.g. [19] where extensive updated literature is given.) The IST methods are quite different in this context and based on analysis on Riemann surfaces.

A comprehensive account of classes of initial data for which the IST is rigorously developed is given in the recent paper [14].

4.2. Well-posedness (WP) of the KdV equation. The interest in WP problems arose almost at the same time as the IST boom started but they are typically approached by means of PDEs techniques [50] (norm estimates, etc.) and the IST is not usually employed. The opposite is quite typical instead: assuming WP one applies the IST method to find the unique solution to KdV. There is also a considerable gap between classes of V_0 's for which WP is established and those V_0 's for which the IST is rigorously justified, the former being much wider than the latter.

Solutions of the KdV can be understood in a number of different senses [50] (classical, strong, weak, etc.) resulting in a variety of different well-posedness results. WP issues are not in the focus of the present paper and we do not attempt to give a comprehensive survey. We mention only the recent sharp results on global well-posedness in $H^{-3/4}(\mathbb{R})$ [23] (which extends [9] where it was proven for⁹ $H^{-s}(\mathbb{R})$ with $s < 3/4$) and a similar result, in the periodic context, [27] where well-posedness is proven in $H^{-1}(\mathbb{T})$. Note that the approach of [27] utilizes complete integrability in a crucial way.

We understand WP in a strong way.

Definition 4.2. *Let $\{V_n(x, t)\}$, $x \in \mathbb{R}$ and $t \geq 0$ be a sequence of classical solutions of (4.1) with the compactly supported initial data*

$$V_n(x, 0) = V_{0,n}(x)$$

approximating $V_0(x)$ in L_{loc}^2 . We call $V(x, t)$, $x \in \mathbb{R}$ and $t \geq 0$, a global natural solution to (4.1), (4.2) if V is a classical solution and

$$V(x, t) = \lim_{n \rightarrow \infty} V_n(x, t)$$

⁸Also referred to as Gelfand-Levitan, Gelfand-Levitan-Marchenko or Faddeev-Marchenko.

⁹We recall $f \in H^s$, $s \in \mathbb{R}$, if $\mathcal{F}f \in L^2(\langle \lambda \rangle^s d\lambda)$.

uniformly on any compact for any $t > 0$ independently of the choice of $\{V_{0,n}\}$.

Our choice of definition is motivated by the methods we employ and it also looks quite natural from the computational and physical point of view.

5. THE REFLECTION COEFFICIENT

In this section we define a right reflection coefficient R from the right incident for arbitrary potentials (i.e. subject to Hypothesis 3.1) on \mathbb{R}_- and L^1 on \mathbb{R}_+ . It is convenient to fragment

$$V(x) = V_-(x) + V_+(x) \quad (5.1)$$

into two potentials $V_\pm(x) = V(x)\chi_\pm(x)$ supported on \mathbb{R}_\pm and consider the reflection coefficient from V_\pm separately and then combine them. Recall that we have agreed to write f_\pm (f could be an operator, space, scattering quantity, m -function, etc.) with \pm if it is associated with \mathbb{R}_\pm .

5.1. Potentials supported on a half line. Assume first that $V = V_-$ is supported on \mathbb{R}_- and subject to Hypothesis 3.1. The Schrödinger equation then has a solution $\Psi(x, \lambda)$ such that for any real λ

$$\Psi(x, \lambda) = \begin{cases} C(\lambda) \Psi_-(x, \lambda) & , \quad x < 0 \\ e^{-i\lambda x} + R_-(\lambda) e^{i\lambda x} & , \quad x \geq 0 \end{cases}$$

where Ψ_- is the Weyl solution and C and R_- are some coefficients¹⁰. Note that Ψ_- turns into the Jost solution ψ_- (3.7) if V is from the Faddeev class. The continuity of $\Psi(x, \lambda)$ and its derivative at $x = 0$ immediately implies that for a.e. real λ

$$R_-(\lambda) = \frac{i\lambda - m_-(\lambda^2 + i0)}{i\lambda + m_-(\lambda^2 + i0)}.$$

Due to the analyticity of m_- and the symmetry property

$$m_-(\bar{z}) = \overline{m_-(z)}$$

the function $R_-(\lambda)$ can be analytically continued into the upper half plane and

$$R_-(-\bar{\lambda}) = \overline{R_-(\lambda)}$$

for any $\lambda \in \mathbb{C}_+$ except for those purely imaginary λ 's for which $\lambda^2 \in \text{Spec}(H_-)$. For real λ 's one can easily see that

$$|R_-(\lambda)| \leq 1.$$

The next important property of R is related to inverse problems. By the Borg-Marchenko uniqueness, m_- determines V and hence R also determines V . Due to analyticity this means that the knowledge of $R(\lambda)$ on any set of real λ 's of positive Lebesgue measure determines $V(x)$ for a.e. $x < 0$. Therefore, no additional information about bound states and their norming constants $\{-\kappa_n^2, c_n\}$ is required in our case. In fact $\{i\kappa_n\}$ are the (simple) poles of R in the upper half plane with residues $\text{Res}(R, i\kappa_n) = ic_n^2$ where c_n are norming constants [4]. For the reader's convenience we summarize what we have said as

¹⁰We remind that in our notation R_- stands for the right reflection coefficient off the potential V_- . In the literature R_- also denotes the left reflection coefficient.

Proposition 5.1. Let $H_- = -\partial_x^2 + V_-(x)$ be the Schrödinger operator on L^2 with V_- supported on \mathbb{R}_- subject to Hypothesis 3.1. Let m_- be the Dirichlet Titchmarsh-Weyl m -function of $-\partial_x^2 + V_-(x)$ corresponding to \mathbb{R}_- . Then the right reflection coefficient $R_-(\lambda)$ is given by

$$R_-(\lambda) = \frac{i\lambda - m_-(\lambda^2)}{i\lambda + m_-(\lambda^2)} = -1 + \frac{2i\lambda}{i\lambda + m_-(\lambda^2)} \quad (5.2)$$

and it represents an analytic in the upper half plane function except for those λ 's on the imaginary line for which $\lambda^2 \in \text{Spec}(H_-)$. Furthermore, it is symmetric with respect to the imaginary axis, i.e.

$$R_-(-\bar{\lambda}) = \overline{R_-(\lambda)}$$

and contractive on the real line:

$$\begin{aligned} |R_-(\lambda)| &\leq 1 \quad \text{for a.e. } \lambda \in \mathbb{R} \\ |R_-(\lambda)| &< 1 \quad \text{for a.e. } \lambda \in \text{Spec}_{ac}(H_-). \end{aligned}$$

The function R may have simple poles $\{i\kappa_n\}$ on the positive part of the imaginary axis. Moreover, the set $\{-\kappa_n^2\}$ coincides with the negative discrete spectrum of H_- and

$$\text{Res}(R, i\kappa_n) = ic_n^2,$$

where c_n is the norming constant corresponding to the bound state $-\kappa_n^2$. If V_- is short range then R_- defined by (5.2) and (3.10) agree.

The statement for the left reflection coefficient L_+ associated with H_+ is almost identical to Proposition 5.1 with

$$L_+(\lambda) = \frac{i\lambda - m_+(\lambda^2)}{i\lambda + m_+(\lambda^2)} = -1 + \frac{2i\lambda}{i\lambda + m_+(\lambda^2)} \quad (5.3)$$

in place of (5.2). We believe that (5.2) is originally due to Faddeev but we could not locate the paper where this appeared first. When V is supported on the whole line, a similar approach was used in [20] and [21] to define certain relative reflection coefficient in situations when there is no classical scattering.

Example 5.2. If $V(x) = -h^2\chi_-(x)$ then $m_-(\lambda^2) = i\sqrt{\lambda^2 + h^2}$ and hence (5.2) takes the form

$$R_-(\lambda) = \frac{\lambda - \sqrt{\lambda^2 + h^2}}{\lambda + \sqrt{\lambda^2 + h^2}} = -\left(\frac{h}{\lambda + \sqrt{\lambda^2 + h^2}}\right)^2$$

which is analytic on $\mathbb{C}_+ \setminus [0, ih]$.

5.2. Potentials supported on the full line. We now define the right reflection coefficient R for any potential subject to Hypothesis 3.1.

Definition 5.3. Let $V = V_- + V_+$ where V_- is arbitrary (i.e. subject to Hypothesis 3.1) and $V_+ \in L^1_+$. Let $V_b := V\chi_{(-b, \infty)}$ and let R_b be the right reflection coefficient. We call the limit

$$R = w - \lim_{b \rightarrow \infty} R_b, \quad (5.4)$$

if it exists, the right reflection coefficient from V .

Lemma 5.4. *The right reflection coefficient $R(\lambda)$ defined by (5.4) is well-defined and satisfies*

$$R = \frac{T_+}{\bar{T}_+} \frac{R_- - \bar{L}_+}{1 - R_- L_+}, \quad (5.5)$$

$$\text{or } R = R_+ + \frac{T_+^2 R_-}{1 - R_- L_+} \quad (5.6)$$

where the subscript \pm indicates that the corresponding scattering quantities are related to V_\pm , the right hand side of (5.6) being independent of a particular partition (5.1). Moreover

$$\begin{aligned} |R(\lambda)| &\leq 1 \text{ for a.e. real } \lambda, \\ |R(\lambda)| &< 1 \text{ for a.e. real } \lambda \in \text{Spec}_{ac}(H) \text{ of multiplicity } 2, \\ \text{and } |R| &= 1 \text{ if and only if } |R_-| = 1. \end{aligned}$$

Proof. To avoid the subscript b we denote $\tilde{V} := V_b = V\chi_{(-b, \infty)}$. The potential splitting

$$\tilde{V} = \tilde{V}_- + V_+$$

implies the fragmentation principle (see, e.g. [4])

$$\begin{pmatrix} 1/\tilde{T} & -\tilde{R}/\tilde{T} \\ \tilde{L}/\tilde{T} & 1/\tilde{T} \end{pmatrix} = \begin{pmatrix} 1/\tilde{T}_- & -\tilde{R}_-/\tilde{T}_- \\ \tilde{L}_-/\tilde{T}_- & 1/\tilde{T}_- \end{pmatrix} \begin{pmatrix} 1/T_+ & -R_+/T_+ \\ L_+/T_+ & 1/\bar{T}_+ \end{pmatrix} \quad (5.7)$$

where each entry is well-defined. Multiplying out the matrices in (5.7)

$$\begin{aligned} \frac{1}{\tilde{T}} &= \frac{1 - L_+ \tilde{R}_-}{T_+ \tilde{T}_-} \\ \frac{\tilde{R}}{\tilde{T}} &= \frac{R_+}{T_+ \tilde{T}_-} + \frac{\tilde{R}_-}{\bar{T}_+ \tilde{T}_-}, \end{aligned}$$

a straightforward algebra yields

$$\tilde{R} = \frac{R_+ + (T_+/\bar{T}_+) \tilde{R}_-}{1 - L_+ \tilde{R}_-}. \quad (5.8)$$

Inserting the following relation from (3.6)

$$\bar{T}_+/T_+ = -\bar{L}_+/R_+$$

into (5.8) yields (5.5). Using the relations (3.5), and (3.6) we have

$$\begin{aligned} \frac{L_+ R_+ - T_+^2}{T_+} &= -\frac{(T_+/\bar{T}_+) R_+ \bar{R}_+ + T_+^2}{T_+} \\ &= -\frac{(T_+/\bar{T}_+) (1 - T_+ \bar{T}_+) + T_+^2}{T_+} = -\frac{1}{\bar{T}_+} \end{aligned}$$

which implies that

$$T_+/\bar{T}_+ = T_+^2 - L_+ R_+.$$

Inserting this relation into (5.8) one obtains

$$\tilde{R} = R_+ + \tilde{G}, \quad \tilde{G} := \frac{T_+^2 \tilde{R}_-}{1 - \tilde{R}_- L_+}. \quad (5.9)$$

As discussed in the previous subsections each quantity in \tilde{G} can admits an analytic continuation into \mathbb{C}_+ and hence \tilde{G} can also be continued into \mathbb{C}_+ . But according to [42], \tilde{m}_- converges uniformly on every compact in \mathbb{C}_+ to m_- as $b \rightarrow \infty$ and hence so does \tilde{R}_- to R . This means that uniformly on every compact in \mathbb{C}_+

$$\lim_{b \rightarrow \infty} \tilde{G} = \frac{T_+^2 R_-}{1 - R_- L_+} =: G. \quad (5.10)$$

(5.10) implies that on the real line $\tilde{R}_- \rightarrow R$ and $\tilde{G} \rightarrow G$ weakly as $b \rightarrow \infty$. Since \tilde{R} in (5.9) is independent of the point of splitting, the reflection coefficient defined by (5.4) is well-defined and (5.6) holds. The last statements of the lemma immediately follow from

$$1 - |R|^2 = \frac{(1 - |R_-|^2)(1 - |L_+|^2)}{|1 - R_- L_+|^2}$$

which in turn follows from the fact that (5.5) represents a Möbius transform (or can be verified by a direct computation). \square

It is quite clear that our definition 5.3 agrees with the standard one if V is short range.

Since each of the scattering quantities in the second term on the right hand side of (5.6) can be analytically extended into the upper half plane, so can the whole second term in (5.6). We have no grounds to believe that R_+ is analytic under the Faddeev condition only. However it is the case if V is reflectionless. Namely, the following curious statement holds.

Proposition 5.5. *Let V be a reflectionless potential (i.e. $R(\lambda) = L(\lambda) = 0$) and $V \in L^1$ then all (left and right) reflection coefficients corresponding to V_{\pm} can be analytically continued into the upper half plane.*

Indeed, it immediately follows from (5.6) that

$$R_+ = -\frac{T_+^2 R_-}{1 - R_- L_+}$$

and hence R_+ admits an analytic continuation into \mathbb{C}_+ . Similarly one proves that L_- has the same property.

Proposition 5.5 is, of course, well known for N -soliton reflectionless potentials but appears to be new as it admits infinitely many solitons.

6. FREDHOLM DETERMINANTS

In this section we specify in what sense we understand Fredholm determinants (Definition 6.1) and prove two important lemmas.

It is well-known that if A is a trace class operator than one can define the invariant¹¹ Fredholm determinant $\det(I + A)$. If A is Hilbert-Schmidt then

$$\det_2(I + A) := \det(I + A) e^{-\text{tr} A}$$

is also well-defined. Apparently if $A \in \mathfrak{S}_1$ then

$$\det(I + A) = \det_2(I + A) \cdot e^{\text{tr} A}.$$

¹¹i.e. independent of a matrix representation of A .

In particular situations it is usually very hard to verify that $A \in \mathfrak{S}_1$. It is not easy even if A is an integral whereas verifying $A \in \mathfrak{S}_2$ merely requires computing a double integral. However for an integral operator on $L^2(S)$, $S \subseteq \mathbb{R}$, with the kernel $A(x, y)$ the condition $A(x, x) \in L^1(S)$ is much easier to check and the trace can then be conveniently defined as the integral of the kernel on the diagonal. Namely, one introduces

Definition 6.1. Let $A : L^2(S) \rightarrow L^2(S)$ be a Hilbert-Schmidt integral operator with the kernel $A(x, y)$. We call A a trace type operator if $A(x, x)$ is well-defined, continuous on S and $A(x, x) \in L^1(S)$ and we set then

$$\begin{aligned} \text{Tr } A & : = \int_S A(x, x) dx \\ \text{Det}(I + A) & : = \det_2(I + A) \cdot e^{\text{Tr } A}. \end{aligned}$$

Of course if $A \in \mathfrak{S}_1$ then

$$\text{Det}(I + A) = \det(I + A)$$

but there are examples of Hilbert-Schmidt integral operators A 's not from \mathfrak{S}_1 but for which $A(x, x) \in L^1(S)$. Such examples are quite pathological though (see [48]).

The Fredholm determinants in two particular cases will be important in our consideration.

Lemma 6.2. Let $\phi(\lambda)$ be a smooth function defined on a piecewise differentiable contour $\Gamma = \{\lambda \in \mathbb{C} : \lambda = \alpha + ih(\alpha), h \geq 0, \alpha \in \mathbb{R}\}$ such that $\frac{\phi(\lambda)}{\text{Im } \lambda} \in L^1(\Gamma)$. Then the integral operator Φ on L_+^2 with the kernel

$$\Phi(x, y) = \int_{\Gamma} e^{i\lambda(x+y)} \phi(\lambda) \frac{d\lambda}{2\pi}, \quad x, y \geq 0,$$

is trace class,

$$\|\Phi\|_{\mathfrak{S}_1} \leq \frac{1}{4\pi} \left\| \frac{\phi(\lambda)}{\text{Im } \lambda} \right\|_{L^1(\Gamma)},$$

and hence $\det(I + \Phi)$ is well-defined in the classical Fredholm sense.

Proof. Set $\gamma(\alpha) := \alpha + ih(\alpha)$. We have $(x, y \geq 0)$

$$\Phi(x, y) = \int e^{i\gamma(\alpha)(x+y)} \phi(\gamma(\alpha)) \gamma'(\alpha) \frac{d\alpha}{2\pi}. \quad (6.1)$$

We now split this integral in a certain way. To this end let $\phi = \phi_1 \phi_2$ be a factorization of ϕ to be chosen later

$$\begin{aligned} & e^{i\gamma(\alpha)(x+y)} \phi(\gamma(\alpha)) \gamma'(\alpha)^{1/2} \\ &= e^{i\gamma(\alpha)x} \phi_1(\gamma(\alpha)) \int e^{i\gamma(\beta)y} \phi_2(\gamma(\beta)) \delta(\beta - \alpha) \gamma'(\beta)^{1/2} d\beta \\ &= \int \left(e^{i\gamma(\alpha)x} \phi_1(\gamma(\alpha)) \int e^{i\gamma(\beta)y} \phi_2(\gamma(\beta)) e^{i(\beta-\alpha)s} \gamma'(\beta)^{1/2} d\beta \right) \frac{ds}{2\pi} \\ &= \phi_1(\gamma(\alpha)) \int e^{i(\gamma(\alpha)x - \alpha s)} \left(\int e^{i(\gamma(\beta)y + \beta s)} \phi_2(\gamma(\beta)) \gamma'(\beta)^{1/2} d\beta \right) \frac{ds}{2\pi} \end{aligned} \quad (6.2)$$

where we have used

$$\delta(\beta - \alpha) = \int e^{i(\beta - \alpha)s} \frac{ds}{2\pi}.$$

Substituting (6.2) into (6.1) yields

$$\begin{aligned} \Phi(x, y) &= \int \left\{ \left(\int e^{i(\gamma(\alpha)x - \alpha s)} \phi_1(\gamma(\alpha)) \gamma'(\alpha)^{1/2} \frac{d\alpha}{2\pi} \right) \right. \\ &\quad \left. \left(\int e^{i(\gamma(\beta)y + \beta s)} \phi_2(\gamma(\beta)) \gamma'(\beta)^{1/2} \frac{d\beta}{2\pi} \right) \right\} ds \\ &= \int \Phi_1(x, s) \Phi_2(s, y) ds \end{aligned} \quad (6.3)$$

where

$$\begin{aligned} \Phi_1(x, s) &= \int \exp \{i(x - s)\alpha - h(\alpha)x\} \phi_1(\gamma(\alpha)) \gamma'(\alpha)^{1/2} d\alpha \\ \Phi_2(s, y) &= \int \exp \{i(s + y)\alpha - h(\alpha)y\} \phi_2(\gamma(\alpha)) \gamma'(\alpha)^{1/2} d\alpha. \end{aligned}$$

By a straightforward computation ($k = 1, 2$)

$$\begin{aligned} \iint_{\mathbb{R}_+} |\Phi_k(x, y)|^2 dx dy &= \int \frac{|\gamma'(\alpha)|}{2h(\alpha)} |\phi_k(\gamma(\alpha))|^2 \frac{d\alpha}{2\pi} \\ &= \int_{\Gamma} \left| \frac{\phi_k(\lambda)}{\sqrt{\operatorname{Im} \lambda}} \right|^2 \frac{|d\lambda|}{4\pi}. \end{aligned}$$

On the other hand

$$\iint_{\mathbb{R}_+} |\Phi_k(x, y)|^2 dx dy = \|\Phi_k\|_{\mathfrak{S}_2}^2$$

and hence

$$\|\Phi_k\|_{\mathfrak{S}_2} = \frac{1}{2\sqrt{\pi}} \left\| \frac{\phi_k(\lambda)}{\sqrt{\operatorname{Im} \lambda}} \right\|_{L^2(\Gamma)}.$$

It follows from (6.3) that $\Phi = \Phi_1 \Phi_2$ and, taking $\phi_1 = \phi_2$ we finally have

$$\begin{aligned} \|\Phi\|_{\mathfrak{S}_1} &\leq \|\Phi_1\|_{\mathfrak{S}_2} \|\Phi_2\|_{\mathfrak{S}_2} = \frac{1}{4\pi} \left\| \sqrt{\frac{\phi(\lambda)}{\operatorname{Im} \lambda}} \right\|_{L^2(\Gamma)}^2 \\ &= \frac{1}{4\pi} \left\| \frac{\phi(\lambda)}{\operatorname{Im} \lambda} \right\|_{L^1(\Gamma)} < \infty \end{aligned}$$

and the lemma is proven. \square

Lemma 6.3. *Let R be as in (3.15) such that $\partial_\lambda R \in L^2$ and $t \geq 0$ is a parameter (time), Then the integral operator Φ on $L^2(a, \infty)$, $a > -\infty$, with the kernel*

$$\Phi(x, y) = \left(\mathcal{F} e^{8i\lambda^3 t} R \right)(x + y)$$

is a trace type operator in the sense of Definition 6.1.

Proof. The fact that $\Phi \in \mathfrak{S}_2$ is well known [11] and, since R is in L^1 , $\Phi(x, x)$ is clearly continuous on \mathbb{R} . We only need to show that $\Phi(x, x)$ decays fast enough

to guarantee $\Phi(x, x) \in L^1(b, \infty)$ for large b 's. For $t = 0$ the statement is obvious and it is enough to assume $t > 0$. By parts

$$\begin{aligned}
2\sqrt{2\pi} \left(\mathcal{F} e^{8i\lambda^3 t} R \right) (2x) &= \int R(\lambda/2) \frac{d e^{i(\lambda x + \lambda^3 t)}}{i(x + 3\lambda^2 t)} \\
&= i \int e^{i(\lambda x + \lambda^3 t)} \partial_\lambda \frac{R(\lambda/2)}{(x + 3\lambda^2 t)} d\lambda \\
&= -i \int e^{i(\lambda x + \lambda^3 t)} \frac{6\lambda t R(\lambda/2)}{(x + 3\lambda^2 t)^2} d\lambda \\
&\quad + i \int e^{i(\lambda x + \lambda^3 t)} \frac{\partial_\lambda R(\lambda/2)}{x + 3\lambda^2 t} d\lambda \\
&=: I_1(2x) + I_2(2x).
\end{aligned}$$

We have

$$\begin{aligned}
\|I_1\|_{L^1(b, \infty)} &\leq 6t \int \left\{ \int_b^\infty \frac{dx}{(x + 3\lambda^2 t)^2} \right\} |\lambda R(\lambda/2)| d\lambda \\
&= 6t \int \frac{|\lambda R(\lambda/2)|}{b + 3\lambda^2 t} d\lambda \leq 12t \|R\|_{L^2} \left\| \frac{\lambda}{b + 3\lambda^2 t} \right\|_{L^2(d\lambda)}.
\end{aligned}$$

Turn now to I_2 . Observe that the following convolution type formula holds¹²:

$$(\mathcal{F} f_x g)(x) = (\mathcal{F} f_x * \mathcal{F} g)(x) \quad (6.4)$$

where the subscript x indicates that f depends on x . Rewriting

$$I_2(2x) = i \int e^{i\lambda x} \left\{ \frac{1}{x + 3\lambda^2 t} \right\} \left\{ e^{i\lambda^3 t} \partial_\lambda R(\lambda/2) \right\} d\lambda$$

and applying the convolution formula (6.4) we have

$$I_2(2x) = i\sqrt{2\pi} F(\cdot, x) * \mathcal{F} \left(e^{i\lambda^3 t} \partial_\lambda R \right) \quad (6.5)$$

where

$$\begin{aligned}
F(s, x) &= \frac{1}{\sqrt{2\pi}} \int \frac{e^{i\lambda s}}{x + 3\lambda^2 t} d\lambda \\
&= \sqrt{2\pi} \frac{e^{-|s|(x/3t)^{1/2}}}{6t (x/3t)^{1/2}}.
\end{aligned}$$

¹²With the usual definition of the convolution

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int f(s) g(x - s) ds.$$

It follows from (6.5) that

$$\begin{aligned}
\|I_2\|_{L^1(b,\infty)} &\leq \int \left\| F(s, x) \mathcal{F} \left(e^{i\lambda^3 t} \partial_\lambda R \right) (x - s) \right\|_{L^1(b,\infty)} ds \\
&\leq \int \|F(s, \cdot)\|_{L^2(b,\infty)} \left\| \mathcal{F} \left(e^{i\lambda^3 t} \partial_\lambda R \right) \right\|_{L^2(b-s,\infty)} ds \\
&\leq \left\| \mathcal{F} \left(e^{i\lambda^3 t} \partial_\lambda R \right) \right\|_{L^2} \int \|F(s, \cdot)\|_{L^2(b,\infty)} ds \\
&= \|\partial_\lambda R\|_{L^2} \int \|F(s, \cdot)\|_{L^2(b,\infty)} ds.
\end{aligned} \tag{6.6}$$

The norm $\|F(s, \cdot)\|_{L^2(b,\infty)}$ on the right hand side of (6.6) can be explicitly evaluated:

$$\begin{aligned}
\|F(s, \cdot)\|_{L^2(b,\infty)} &= \frac{\sqrt{2\pi}}{6t} \left(\int_b^\infty \frac{e^{-2|s|(x/3t)^{1/2}}}{x/3t} dx \right)^{1/2} \\
&= \sqrt{\frac{\pi}{6t}} \left(\int_b^\infty \frac{e^{-2|s|(x/3t)^{1/2}}}{x} dx \right)^{1/2} \\
&= \sqrt{\frac{\pi}{3t}} \left(\int_{2\sqrt{\frac{b}{3t}|s|}}^\infty \frac{e^{-x}}{x} dx \right)^{1/2},
\end{aligned}$$

which immediately implies that $\|F(s, \cdot)\|_{L^2(b,\infty)}$ is continuous with respect to $s \in \mathbb{R} \setminus \{0\}$ and

$$\|F(s, \cdot)\|_{L^2(b,\infty)} = o \left(e^{-\sqrt{\frac{b}{3t}}|s|} \right), \quad s \rightarrow \pm\infty. \tag{6.7}$$

Around $s = 0$ (denoting $\alpha := 2\sqrt{\frac{b}{3t}}|s|$)

$$\int_\alpha^\infty \frac{e^{-x}}{x} dx = \frac{e^{-\alpha}}{\alpha} - \int_\alpha^\infty \frac{e^{-x}}{x^2} dx = O(1/\alpha) = O(1/|s|), \quad s \rightarrow 0,$$

and therefore

$$\|F(s, \cdot)\|_{L^2(b,\infty)} = O(1/|s|^{1/2}), \quad s \rightarrow 0. \tag{6.8}$$

(6.7) and (6.8) imply that $\int \|F(s, \cdot)\|_{L^2(b,\infty)} ds$ is finite and the lemma is proven since $\|\partial_\lambda R\|_{L^2}$ is also finite. \square

7. A HANKEL INTEGRAL OPERATOR

In this section we introduce and study a Hankel integral operator particularly important in the context of the IST.

Definition 7.1. Let μ be a non-negative finite measure on \mathbb{R}_+ and ϕ be an L^∞ function. We call an operator $\mathbb{M} : L_+^2 \rightarrow L_+^2$ a Marchenko type operator associated with (μ, ϕ) if

$$\mathbb{M} = \mathbb{M}_1 + \mathbb{M}_2 \tag{7.1}$$

where \mathbb{M}_1 is the integral operator with the kernel

$$M_1(x, y) = \int_{\mathbb{R}_+} e^{-\alpha(x+y)} d\mu(\alpha) \tag{7.2}$$

and $(\chi := \chi_+)$

$$\mathbb{M}_2 = \chi \mathcal{F} \phi \mathcal{F}. \tag{7.3}$$

Here ϕ and χ are the operators of multiplication by the functions ϕ and χ respectively.

The Marchenko operator $\mathbb{M}_{x,t}$ defined by (4.5) – (4.6) is Marchenko type as it can be represented by (7.1) – (7.3) with

$$\begin{aligned} d\mu(\alpha) &= \sum_{n=1}^N c_n^2 e^{-2\alpha x + 8\alpha^3 t} \delta(\alpha - \kappa_n) d\alpha, \\ \phi(\lambda) &= e^{2i\lambda x + 8i\lambda^3 t} R(\lambda), \end{aligned}$$

where δ denotes the Dirac delta function.

The operator \mathbb{M} is clearly a Hankel operator. In this section we are concerned with two main questions: when is \mathbb{M} a trace class operator (or at least when is $\text{Det}(I + \mathbb{M})$ well-defined) and when is $I + \mathbb{M}$ boundedly invertible?

Introduce yet another two parametric family ($z \in \mathbb{C}$ and $t \geq 0$ are parameters) of integral operators

$$(\mathbb{G}_{z,t}f)(x) := \int_{\mathbb{R}_+} G_{z,t}(x, y) f(y) dy, \quad (7.4)$$

acting in L_+^2 with the kernel $G_{z,t}(x, y)$ defined by

$$G_{z,t}(x, y) := \int_{\Gamma} e^{i\lambda(x+y)} g_{z,t}(\lambda) \frac{d\lambda}{2\pi} \quad (7.5)$$

where Γ is as in the Figure 1 and

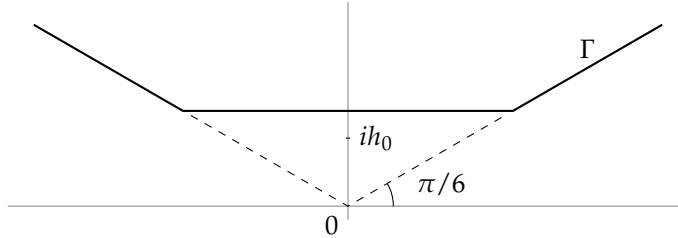


FIGURE 1. Contour of integration Γ in equation (7.5), where $h_0 = |\inf \text{Spec}(H)|^{1/2}$.

$$g_{z,t}(\lambda) := e^{2i\lambda z} e^{8i\lambda^3 t} G(\lambda) \quad (7.6)$$

with some function G specified in the proposition below. The operator $\mathbb{G}_{z,t}$ is a Hankel operator having some important properties which we summarize in the following statement.

Proposition 7.2. *Let $\mathbb{G}_{z,t} : L_+^2 \rightarrow L_+^2$ be defined by (7.4)-(7.5) with some G analytic in $\mathbb{C}_+ \setminus [0, ia]$, $a \geq 0$, subject to*

- (i)
- (ii) (symmetry)

$$G(-\bar{\lambda}) = \overline{G(\lambda)} \quad (7.7)$$

- (iii) (decay)

$$|G(\lambda)| \rightarrow 0, |\lambda| \rightarrow \infty, 0 < \arg \lambda < \pi$$

(iv) (boundary values on the real line)

$$G(\lambda + i0) \in L^\infty$$

(v) (boundary behavior on the imaginary line)

$$d\rho(\alpha) := \frac{1}{\pi} \operatorname{Im} G(+0 + i\alpha) d\alpha \quad (7.8)$$

defines a non-negative finite on $[0, a]$ measure. I.e.

$$d\rho(\alpha) \geq 0 \text{ and } \int_0^a d\rho(\alpha) < \infty.$$

Then

(1) $G_{z,t}$ is a Marchenko type operator (Definition 7.1) associated with (μ, ϕ) given by

$$d\mu(\alpha) = e^{-2\alpha z + 8\alpha^3 t} d\rho(\alpha) \text{ and } \phi = g_{z,t}.$$

Moreover

(2) $G_{z,t}$ is selfadjoint for any $z \in \mathbb{R}$ and $t \geq 0$

(3) $G_{z,t} \in \mathfrak{S}_1$ for any $z \in \mathbb{C}$ and $t > 0$ and

$$\|G_{z,t}\|_{\mathfrak{S}_1} \leq \frac{1}{4\pi} \left\| \frac{g_{z,t}(\lambda)}{\operatorname{Im} \lambda} \right\|_{L^1(\Gamma)}$$

(4) $G_{z,t}$ is entire and $(I + G_{z,t})^{-1}$ is a meromorphic operator valued function in z on the entire complex plane for any $t > 0$.

Proof. Due to (7.7) the kernel (7.5) is real for real z and symmetric. The operator $G_{z,t}$ is therefore selfadjoint for real z and Part (2) is proven. To prove the representation (7.1) – (7.3) one merely needs to deform the contour to the real line. The only issue is to make sure that the corresponding integral operators converge strongly. Denoting $(\Gamma^+ := \cap \{\operatorname{Re} z > 0\})$

$$G_{z,t}(x) := \operatorname{Re} \int_{\Gamma^+} e^{i\lambda x} g_{z,t}(\lambda) \frac{d\lambda}{\pi},$$

we have

$$\begin{aligned} G_{z,t}(x) &= G_{z,t}^{(1)}(x) + G_{z,t}^{(2)}(x) + G_{z,t}^{(3)}(x) + G_{z,t}^{(4)}(x) \\ G_{z,t}^{(k)}(x) &:= \frac{1}{\pi} \operatorname{Re} \int_{\gamma_k} e^{i\lambda x} G_{z,t}(\lambda) \frac{d\lambda}{\pi}, \quad k = 1, 2, 3, 4. \end{aligned} \quad (7.9)$$

where

$$\begin{aligned} \gamma_1 &= \{i\alpha + 0 : 0 < \alpha < a\}, & \gamma_2 &= (0, N), \\ \gamma_3 &= \{Ne^{i\theta}, 0 < \theta < \pi/6\}, & \gamma_4 &= \{re^{i\pi/6} : r > N\} \end{aligned}$$

as shown in Figure 2

The representation (7.9) leads to

$$G_{z,t} = G_{z,t}^{(1)} + G_{z,t}^{(2)} + G_{z,t}^{(3)} + G_{z,t}^{(4)}$$

where $G_{z,t}^{(k)}$ are the integral operators defined by $(f_+ = f\chi_+)$

$$G_{z,t}^{(k)} f(x) = \int G_{z,t}^{(k)}(x+y) f_+(y) dy.$$

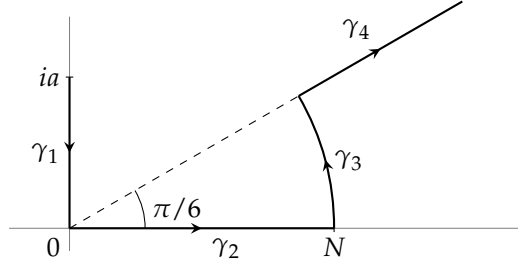


FIGURE 2.

The arbitrary $N > 0$ will be taken to infinity. For $G_{z,t}^{(1)}(x)$ we get

$$\begin{aligned}
 G_{z,t}^{(1)}(x) &= \frac{1}{\pi} \lim_{\varepsilon \rightarrow +0} \operatorname{Re} \int_a^0 e^{i(\varepsilon+i\alpha)x} g_{z,t}(\varepsilon+i\alpha) i d\alpha \\
 &= \frac{1}{\pi} \lim_{\varepsilon \rightarrow +0} \int_0^a e^{-\alpha x} \operatorname{Im} g_{z,t}(\varepsilon+i\alpha) d\alpha \\
 &= \frac{1}{\pi} \lim_{\varepsilon \rightarrow +0} \int_0^a e^{-\alpha x} e^{-2\alpha z + 8\alpha^3 t} \operatorname{Im} G(\varepsilon+i\alpha) d\alpha \\
 &= \int_0^a e^{-\alpha x} e^{-2\alpha z + 8\alpha^3 t} d\rho(\alpha),
 \end{aligned}$$

and $G_{z,t}^{(1)}$ hence produces M_1 in the decomposition (7.1). For $G_{z,t}^{(2)}: (f_+ = \chi_+ f)$

$$\begin{aligned}
 G_{z,t}^{(2)} f(x) &= \int G_{z,t}^{(2)}(x+y) f_+(y) dy \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-N}^N e^{i\lambda x} (g_{z,t} \mathcal{F} f_+)(\lambda) d\lambda.
 \end{aligned}$$

Since $\mathcal{F} f_+ \in L^2$ and $g_{z,t} \in L^\infty$ we have $g_{z,t} \mathcal{F} f_+ \in L^2$, and hence

$$\begin{aligned}
 \|G_{z,t} f - G_{z,t}^{(2)} f\|_{L^2} &= \left\| \frac{1}{\sqrt{2\pi}} \int \left(1 - \chi_{(-N,N)}(\lambda)\right) e^{i\lambda x} g_{z,t}(\lambda) (\mathcal{F} f_+)(\lambda) d\lambda \right\|_{L^2} \\
 &\lesssim \left\| \int e^{i\lambda x} \left(1 - \chi_{(-N,N)}(\lambda)\right) g_{z,t}(\lambda) (\mathcal{F} f_+)(\lambda) d\lambda \right\|_{L^2} \\
 &= \left\| \mathcal{F} \left(1 - \chi_{(-N,N)}\right) g_{z,t}(\mathcal{F} f_+) \right\|_{L^2} \\
 &= \left\| \left(1 - \chi_{(-N,N)}\right) g_{z,t} \mathcal{F}(f_+) \right\|_{L^2} \rightarrow 0, \quad N \rightarrow \infty.
 \end{aligned}$$

Therefore $G_{z,t}^{(2)} \rightarrow G_{z,t}$ in the strong operator topology for any real z and t . It follows from Lemma 6.2 that $\|G_{z,t}^{(4)}\|_{\mathfrak{S}_2} \rightarrow 0$ when $N \rightarrow \infty$. It remains to show that $\|G_{z,t}^{(3)} f\|_{L^2} \rightarrow 0$, $N \rightarrow \infty$. Since $G_{z,t}$ and $G_{z,t}^{(1)}$ are independent of N and, as we have already proven, $G_{z,t}^{(2)} + G_{z,t}^{(4)}$ strongly converges to $G_{z,t}^+$, it follows from the decomposition (7.9) that $G_{z,t}^{(3)}$ must also converge strongly. It is sufficient to show that $G_{z,t}^{(3)}$ converges weakly to 0 as $N \rightarrow \infty$ as this will force the strong convergence

of $\mathbb{G}_{z,t}^{(3)}$ to 0. We have ($\xi = e^{i\theta}$)

$$\begin{aligned} (\mathbb{G}_{z,t}^{(3)} f)(x) &= \operatorname{Re} \int_0^\infty \left(\int_0^{\pi/6} e^{iN\xi(x+y)} g_{z,t}(N\xi) iN\xi \frac{d\theta}{\pi} \right) f(y) dy \quad (7.10) \\ &= -N \operatorname{Im} \int_0^{\pi/6} \left(\int_0^\infty e^{iN\xi y} f(y) dy \right) e^{iN\xi x} g_{z,t}(N\xi) \xi \frac{d\theta}{\pi} \\ &= -\sqrt{2}N \operatorname{Im} \int_0^{\pi/6} e^{iN\xi x} (g_{z,t} \mathcal{F} f_+)(N\xi) \xi \frac{d\theta}{\sqrt{\pi}} \end{aligned}$$

Changing the order of integration in (7.10) is justified as Fubini's theorem clearly applies. Consider $\langle \mathbb{G}_{z,t}^{(3)} f, \varphi \rangle$ with arbitrary $f, \varphi \in L^2$ which, without loss of generality, can be taken real. Hence one has

$$\begin{aligned} \left| \langle \mathbb{G}_{z,t}^{(3)} f, \varphi \rangle \right| &= \sqrt{2}N \left| \int_0^{\pi/6} \operatorname{Im} (g_{z,t}) (\mathcal{F} f_+) \mathcal{F} (\varphi_+) (N\xi) \xi d\theta \right| \quad (7.11) \\ &\lesssim N \int_0^{\pi/6} |(g_{z,t} \mathcal{F} f_+ \mathcal{F} \varphi_+) (N\xi)| d\theta \end{aligned}$$

Since $\mathcal{F} f_+, \mathcal{F} \varphi_+$ are both in H_+^2 their product $F := (\mathcal{F} f_+) (\mathcal{F} \varphi_+)$ is in H_+^1 and so by the Hardy-Littlewood theorem the maximal function $F^*(N) := \sup_{a>0} |F(Ne^{ia})|$ is in L^1 . Therefore it follows from (7.11) that

$$\begin{aligned} \left| \langle \mathbb{G}_{z,t}^{(3)} f, \varphi \rangle \right| &\lesssim NF^*(N) \int_0^{\pi/6} |g_{z,t}(N\xi)| d\theta \\ &\lesssim NF^*(N) \int_0^{\pi/6} \exp \left\{ \frac{4}{\pi} N (|z| - 12N^2 t) \theta \right\} |g(N\xi)| d\theta \\ &\lesssim \frac{F^*(N)}{12N^2 t - |z|}. \end{aligned}$$

The latter implies that there is a sequence $\{N_k\} \rightarrow \infty$ such that $\langle \mathbb{G}_{z,t}^{(3)} f, \varphi \rangle \rightarrow 0$ and Part (1) is finally proven.

Part (3), immediately follows from Lemma 6.2. It only remains to show Part (4). To this end consider for any f and g from L_+^2 the function $\langle \mathbb{G}_{z,t} f, g \rangle$ which is clearly differentiable in z for any complex z and $t > 0$ and hence $\mathbb{G}_{z,t}$ is an entire operator-valued function. We now show that for any $t > 0$ the operator $I + \mathbb{G}_{z,t}$ is boundedly invertible for at least one real z . Consider

$$\begin{aligned} \langle \mathbb{G}_{z,t} f, f \rangle &= \int_{\mathbb{R}_+} \overline{f(x)} dx \int_{\mathbb{R}_+} dy G_{z,t}(x, y) f(y) \\ &= \int_{\Gamma} g_{z,t}(\lambda) \left\{ \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i\lambda x} \overline{f(x)} dx \right\} \left\{ \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i\lambda y} f(y) dy \right\} d\lambda \\ &= \int_{\Gamma} g_{z,t}(\lambda) \mathcal{F} \overline{f_+}(\lambda) \mathcal{F} f_+(\lambda) d\lambda = \int_{\Gamma} g_{z,t}(\lambda) F(\lambda) d\lambda, \end{aligned}$$

where $F := \overline{\mathcal{F}f_+} \mathcal{F}f_+$. Notice that since $\overline{\mathcal{F}f_+}, \mathcal{F}f_+ \in H_+^2$, the function $F \in H_+^1$ and therefore by (2.1)

$$\begin{aligned} |F(\lambda)| &\leq \frac{\|F\|_{H_+^1}}{\operatorname{Im} \lambda} \leq \frac{\|\mathcal{F}f_+\|_{H_+^2} \|\overline{\mathcal{F}f_+}\|_{H_+^2}}{\operatorname{Im} \lambda} \\ &= \frac{\|f\|_{L_+^2} \|\overline{f}\|_{L_+^2}}{\operatorname{Im} \lambda} = \frac{\|f\|_{L_+^2}^2}{\operatorname{Im} \lambda}. \end{aligned}$$

Hence for $z > 0$

$$\begin{aligned} |\langle \mathbf{G}_{z,t} f, f \rangle| &\leq \int_{\Gamma} |g_{z,t}(\lambda)| |F(\lambda)| |d\lambda| \\ &\leq \|f\|_{L_+^2}^2 \int_{\Gamma} |g_{z,t}(\lambda)| \frac{|d\lambda|}{\operatorname{Im} \lambda} \\ &\leq \|f\|_{L_+^2}^2 \sup_{\lambda \in \Gamma} |e^{2i\lambda z}| \int_{\Gamma} |g_{0,t}(\lambda)| \frac{|d\lambda|}{\operatorname{Im} \lambda} \\ &= \|f\|_{L_+^2}^2 \sup_{\lambda \in \Gamma} e^{-2z \operatorname{Im} \lambda} \int_{\Gamma} |g_{0,t}(\lambda)| \frac{|d\lambda|}{\operatorname{Im} \lambda} \\ &\leq \|f\|_{L_+^2}^2 e^{-2zh} \int_{\Gamma} |g_{0,t}(\lambda)| \frac{|d\lambda|}{\operatorname{Im} \lambda}, \end{aligned} \quad (7.12)$$

where $h := \inf_{\lambda \in \Gamma} \operatorname{Im} \lambda$. If we choose (the integral $\int_{\Gamma} |g_{0,t}(\lambda)| \frac{|d\lambda|}{\operatorname{Im} \lambda}$ is apparently finite for any $t \geq 0$)

$$z = z_0 := \frac{1}{2h} \left| \ln 2 \int_{\Gamma} |g_{0,t}(\lambda)| \frac{|d\lambda|}{\operatorname{Im} \lambda} \right|, \quad (7.13)$$

then

$$|\langle \mathbf{G}_{z_0,t} f, f \rangle| \leq \frac{1}{2} \|f\|_{L_+^2}^2$$

and it now follows from (7.12) that

$$\langle (I + \mathbf{G}_{z_0,t}) f, f \rangle \geq \frac{1}{2} \|f\|_{L_+^2}^2$$

which shows that for any $t \geq 0$ there is z_0 found by (7.13) such that $I + \mathbf{G}_{z_0,t}$ is invertible. Therefore by [49] $(I + \mathbf{G}_{z,t})^{-1}$ is a meromorphic (operator) function of z on the whole complex plane for any $t > 0$.¹³ The proposition is proven. \square

Lemma 7.3. *Let \mathbb{M} defined by (7.1) – (7.3) be compact. Then the operator $I + \mathbb{M}$ is boundedly invertible if at least one of the following conditions holds:*

- (1) $\mu(S) > 0$ for some set S of non-uniqueness of an H_+^2 function
- (2) $|\phi(\lambda)| < 1$ a.e. on a set $S \subset \mathbb{R}$ of positive Lebesgue measure.

Proof. The proof is standard. Since the operator \mathbb{M} is compact, the point -1 may only be its eigenvalue. One needs to show that it is not the case. Consider the homogeneous equation

$$f + \mathbb{M}f = 0. \quad (7.14)$$

Denoting $f_+ = \chi_+ f$, equation (7.14) therefore implies

$$\langle f_+, f_+ \rangle + \langle \mathbb{M}f_+, f_+ \rangle = 0$$

¹³for $t = 0$ the operator $\mathbb{M}_{z,0}$ need not of course be analytic.

or explicitly

$$\int |f_+(x)|^2 dx + \int_{\mathbb{R}_+} \left| \int e^{-\alpha x} f_+(x) dx \right|^2 d\mu(\alpha) + \int \phi(\lambda) \mathcal{F}f_+(\lambda) \overline{\mathcal{F}f_+(\lambda)} d\lambda = 0.$$

Assuming that $\|f_+\|_{L^2} = 1$ and observing that $\int e^{-\alpha x} f_+(x) dx = \sqrt{2\pi} \mathcal{F}f_+(i\alpha)$ the last equation takes the form

$$1 + 2\pi \int_{\mathbb{R}_+} |\mathcal{F}f_+(i\alpha)|^2 d\mu(\alpha) + \operatorname{Re} \int \phi(\lambda) \mathcal{F}f_+(\lambda) \overline{\mathcal{F}f_+(\lambda)} d\lambda = 0.$$

It follows from this equation that $\left(\|\mathcal{F}\overline{f_+}\|_{L^2} = 1\right)$

$$\begin{aligned} & -2\pi \int_{\mathbb{R}_+} |\mathcal{F}f_+(i\alpha)|^2 d\mu(\alpha) \\ &= 1 + \operatorname{Re} \int \phi(\lambda) \mathcal{F}f_+(\lambda) \overline{\mathcal{F}f_+(\lambda)} d\lambda \\ &\geq 1 - \left| \int \phi(\lambda) \mathcal{F}f_+(\lambda) \overline{\mathcal{F}f_+(\lambda)} d\lambda \right| \\ &\geq 1 - \int |\phi(\lambda)| |\mathcal{F}f_+(\lambda)| |\overline{\mathcal{F}f_+(\lambda)}| d\lambda \\ &\geq 1 - \|\phi \mathcal{F}f_+\|_{L^2}. \end{aligned}$$

Thus

$$\|\phi \mathcal{F}f_+\|_{L^2} \geq 1 + 2\pi \int_{\mathbb{R}_+} |\mathcal{F}f_+(i\alpha)|^2 d\mu(\alpha). \quad (7.15)$$

If $|\phi(\lambda)| < 1$ a.e. on a set S of positive Lebesgue measure then $\|\phi \mathcal{F}f_+\|_{L^2} < \|\mathcal{F}f_+\|_{L^2} = 1$ (as $\mathcal{F}f_+$ is in H_+^2 and hence cannot vanish on S) and (7.15) implies the obvious contradiction

$$0 \leq \int_{\mathbb{R}_+} |\mathcal{F}f_+(i\alpha)|^2 d\mu(\alpha) < 0.$$

Therefore $f_+ = 0$ and $I + \mathbb{M}$ is boundedly invertible. Assume now that Condition 1 is satisfied. Without loss of generality we may assume $|\phi(\lambda)| = 1$ a.e. on \mathbb{R} . Then $\|\phi \mathcal{F}f_+\|_{L^2} = \|\mathcal{F}f_+\|_{L^2} = 1$ and (7.15) implies

$$\int_{\mathbb{R}_+} |\mathcal{F}f_+(i\alpha)|^2 d\mu(\alpha) \leq 0$$

forcing $\mathcal{F}f_+(i\alpha) = 0$ for every $\alpha \in S$. But $\mathcal{F}f_+$ is an H_+^2 function and hence cannot vanish on S . The lemma is proven. \square

8. MAIN RESULTS

In this section we present our main results (given in two statements) which will appear as simple consequences of the considerations above.

8.1. The properties of the Marchenko operator. The following statement relates the properties of a Marchenko type operator with the properties of the underlying potential.

Theorem 8.1. *Let V be a real function such that*

$$\inf \operatorname{Spec} \left(-\partial_x^2 + V \right) = -h_0 > -\infty$$

and that V admits a decomposition

$$V = V_- + V_+ \quad (V_{\pm} = \chi_{\pm} V) \quad (8.1)$$

where V_- is arbitrary (subject to Hypothesis 3.1 at $-\infty$) and $V_+ \in L_+^1(\langle x \rangle dx)$. Consider a two parametric family of Marchenko type (Definition 7.1) operators $\mathbb{M}_{z,t}$ ($z \in \mathbb{C}$ and $t \geq 0$) associated with the data $(\rho_{z,t}, R_{z,t})$ where $d\rho_{z,t}(\alpha) := e^{-2\lambda z + 8\alpha^3 t} d\rho(\alpha)$, and $d\rho(\alpha)$ is defined by

$$d\rho(\alpha) = \left\{ \sum_{n=1}^N (c_n^+)^2 \delta(\alpha - \kappa_n^+) + \left(\frac{T_+^2 \operatorname{Im} R_-}{|1 - R_- L_+|^2} \right) (+0 + i\alpha) \right\} d\alpha, \quad (8.2)$$

and

$$R_{z,t}(\lambda) := e^{i(2\lambda z + 8\lambda^3 t)} R(\lambda).$$

The measure ρ is non-negative, finite, supported on $[0, h_0]$ and independent of the choice of the splitting point in (8.1). The operator $\mathbb{M}_{z,t}$ therefore is well defined and has the following properties:

- (1) $\mathbb{M}_{z,t}$ is selfadjoint and bounded for any real z and $t \geq 0$.
- (2) If $V_+ \in L_+^2(e^{\delta x^{1/2}} dx)$ for some $\delta > 0$ then $\mathbb{M}_{z,t}$ is a trace type real analytic operator-valued function in z for any $t > 0$ and $(I + \mathbb{M}_{z,t})^{-1}$ is real meromorphic¹⁴.
- (3) If $V_+ = 0$ then $\mathbb{M}_{z,t}$ is an entire in z operator-valued function of trace class for any $t > 0$ and $(I + \mathbb{M}_{z,t})^{-1}$ is meromorphic in z .
- (4) If $\operatorname{Spec}_{\text{ac}}(-\partial_x^2 + V)$ has a nonempty component of multiplicity two then $I + \mathbb{M}_{z,t}$ is boundedly invertible for any real z and $t \geq 0$.

Proof. Prove first that $\rho(\alpha)$ defined by (8.2) is independent of the particular decomposition (8.1). Take \tilde{V} from the proof of Lemma 5.4. Since $\tilde{V} \in L^1(\langle x \rangle dx)$ the measure $\tilde{\rho}(\alpha)$ can be alternatively computed by

$$d\tilde{\rho}(\alpha) = \sum_{n=1}^{\tilde{N}} \tilde{c}_n^2 \delta(\alpha - \tilde{\kappa}_n) d\alpha$$

and hence it is independent of the splitting point in (8.1). On the other hand,

$$d\tilde{\rho}(\alpha) = \left\{ \sum_{n=1}^N (c_n^+)^2 \delta(\alpha - \kappa_n^+) + \left(\frac{T_+^2 \operatorname{Im} \tilde{R}_-}{|1 - \tilde{R}_- L_+|^2} \right) (i\alpha + 0+) \right\} d\alpha. \quad (8.3)$$

¹⁴I.e. a ratio of two real analytic functions.

Note that T_+ and L_+ are real on the imaginary line

$$\begin{aligned}
\left(\frac{T_+^2 \operatorname{Im} \tilde{R}_-}{|1 - \tilde{R}_- L_+|^2} \right) (+0 + i\alpha) &= T_+^2 (+0 + i\alpha) \operatorname{Im} \left(\frac{\tilde{R}_- - |\tilde{R}_-|^2 \overline{L_+}}{|1 - \tilde{R}_- L_+|^2} \right) (+0 + i\alpha) \\
&= T_+^2 (+0 + i\alpha) \operatorname{Im} \left(\frac{\tilde{R}_-}{1 - \tilde{R}_- L_+} \right) (+0 + i\alpha) \\
&= \operatorname{Im} \left(\frac{\tilde{R}_-}{1 - \tilde{R}_- L_+} T_+^2 \right) (+0 + i\alpha) \\
&= \operatorname{Im} \tilde{G} (+0 + i\alpha)
\end{aligned} \tag{8.4}$$

where \tilde{G} is defined by (5.9). Inserting (8.4) into (8.3) we have

$$d\tilde{\rho}(\alpha) = \left\{ \sum_{n=1}^N (c_n^+)^2 \delta(\alpha - \kappa_n^+) + \operatorname{Im} \tilde{G} (+0 + i\alpha) \right\} d\alpha.$$

As proven in Lemma 5.4,

$$\begin{aligned}
\lim_{b \rightarrow \infty} \operatorname{Im} \tilde{G} (+0 + i\alpha) d\alpha &= \operatorname{Im} G (+0 + i\alpha + 0) d\alpha \\
&= \left(\frac{T_+^2 \operatorname{Im} R_-}{|1 - R_- L_+|^2} \right) (+0 + i\alpha) d\alpha
\end{aligned}$$

and therefore $\lim_{b \rightarrow \infty} d\tilde{\rho}(\alpha) = d\rho(\alpha)$. But each $\tilde{\rho}$ is independent of the split in (8.1) and thus ρ doesn't depend on (8.1). We prove now that $\rho(\alpha)$ is a non-negative finite measure on $[0, h_0]$. Non-negativity follows from (8.4) (with the tilde dropped and using (5.2))

$$\begin{aligned}
\operatorname{Im} G (+0 + i\alpha) &= \operatorname{Im} \left(\frac{R_-}{1 - R_- L_+} T_+^2 \right) (+0 + i\alpha) \\
&= 2\alpha \left(\frac{T_+^2}{|1 - R_- L_+|^2 |i\lambda + m_-|^2} \right) (+0 + i\alpha) \operatorname{Im} m_- (-\alpha^2 + i0 +) \\
&\geq 0.
\end{aligned}$$

We show now that $\rho(\alpha)$ is a finite measure on $[0, h_0]$. Choose, for simplicity, the splitting point so that $-\partial_x^2 + V_-$ has only one bound state $-\kappa_0^2$ and evaluate (8.2) around $\alpha = \kappa_0 \in [0, h_0]$ and $\alpha = 0$ separately. From (5.2) and (5.3) one has

$$\begin{aligned}
G &= \frac{R_-}{1 - R_- L_+} T_+^2 \\
&= \left(1 - \frac{i\lambda - m_-}{i\lambda + m_-} \frac{i\lambda - m_+}{i\lambda + m_+} \right)^{-1} \frac{i\lambda - m_-}{i\lambda + m_-} T_+^2 \\
&= \left(-1 + \frac{i\lambda + m_+}{m_- + m_+} \right) \frac{i\lambda + m_+}{2i\lambda} T_+^2 \\
&= \left(-1 + \frac{i\lambda + m_+}{m_- + m_+} \right) \frac{2i\lambda}{i\lambda + m_+} g \\
&= \frac{2i\lambda}{m_- + m_+} g - \frac{2i\lambda}{i\lambda + m_+} g,
\end{aligned} \tag{8.5}$$

where

$$g := \left(\frac{i\lambda + m_+}{2i\lambda} T_+ \right)^2.$$

It follows from (5.3) that

$$\frac{i\lambda + m_+ (\lambda^2)}{2i\lambda} = (1 + L_+ (\lambda))^{-1} \quad (8.6)$$

and hence

$$g(i\alpha) = \left(\frac{T_+(i\alpha)}{1 + L_+(i\alpha)} \right)^2.$$

Since T_+ and L_+ both have a simple pole at $\lambda = i\kappa_0$, the function g has a removable singularity at $\lambda = i\kappa_0$. Moreover, due to the symmetry, $T_+(i\alpha)$ and $L_+(i\alpha)$ are both real and hence $g(i\alpha) \geq 0$ and bounded away from $\alpha = 0$.

Note that since $m_\pm(z)$ and $i\sqrt{z}$ are both Herglotz, i.e. $\mathbb{C}_+ \rightarrow \mathbb{C}_+$, we immediately conclude that $-(m_-(z) + m_+(z))^{-1}$ and $-(m_+(z) + i\sqrt{z})^{-1}$ are also Herglotz and hence admit a Herglotz representation similar to (3.4) with some non-negative finite measures μ_1 and μ_2 respectively computed by

$$\begin{aligned} d\mu_1(s) &= -\frac{1}{\pi} \operatorname{Im} (m_-(s + i0+) + m_+(s + i0+))^{-1} ds. \\ d\mu_2(s) &= -\frac{1}{\pi} \operatorname{Im} (m_+(s + i0+) + i\sqrt{s})^{-1} ds. \end{aligned}$$

Therefore it follows from (7.8) and (8.5) that

$$\begin{aligned} d\rho(\alpha) &= \frac{1}{\pi} g(i\alpha) \left\{ -\operatorname{Im} \left(m_-(-\alpha^2 + i0+) + m_+(-\alpha^2 + i0+) \right)^{-1} (-2\alpha) \right. \\ &\quad \left. + \operatorname{Im} \left(m_+(-\alpha^2 + i0+) - \alpha \right)^{-1} (-2\alpha) \right\} d\alpha \\ &= g(i\alpha) (d\mu_1(-\alpha^2) - d\mu_2(-\alpha^2)) \\ &= g(i\alpha) (-d\rho_1(\alpha) + d\rho_2(\alpha)), \end{aligned}$$

where $d\rho_k(\alpha) := -d\mu_k(-\alpha^2)$ $k = 1, 2$ and finite (non-negative) measures. Since, as already proven, $g(i\alpha)$ is bounded away from $\alpha = 0$ the measure $\rho(\alpha)$ is real and finite on $[\varepsilon, h_0]$, $\varepsilon > 0$. If $\alpha = 0$ is an exceptional point then $\lim_{\alpha \rightarrow +0} L_+(i\alpha) > -1$ and $\lim_{\alpha \rightarrow +0} g(i\alpha)$ is finite and $\rho(\alpha)$ is non-negative and finite on $[0, h_0]$. It remains to show that $\rho(\alpha)$ is finite on $[0, h_0]$ even if $\lim_{\alpha \rightarrow +0} L_+(i\alpha) = -1$, i.e. $\alpha = 0$ is a generic point. To this end we need to represent G differently:

$$\begin{aligned} G &= 2i\lambda \frac{(i\lambda - m_-)(i\lambda + m_+)}{m_- + m_+} \left(\frac{T_+}{2i\lambda} \right)^2 \\ &= 2i\lambda \left\{ \frac{(i\lambda)^2}{m_- + m_+} - \frac{m_- m_+}{m_- + m_+} - i\lambda + \frac{2i\lambda m_+}{m_- + m_+} \right\} \omega \\ &= G_1 + G_2 + G_3 + G_4, \quad \omega := \left(\frac{T_+}{2i\lambda} \right)^2. \end{aligned}$$

If $\alpha = 0$ is a generic point $\frac{T_+(i\alpha)}{\alpha}$ remains bounded as $\alpha \rightarrow +0$ and hence so does $\omega(i\alpha)$.

The term G_3 is trivial:

$$\frac{1}{\pi} \operatorname{Im} G_3(+0 + i\alpha) d\alpha = -\frac{2}{\pi} \alpha^2 \omega(i\alpha) d\alpha.$$

Since $-(m_- + m_+)^{-1}$ and $\frac{m_- m_+}{m_- + m_+} = \frac{1}{1/m_- + 1/m_+}$ are Herglotz, by the same arguments as above, one can easily conclude that the measures

$$\frac{1}{\pi} \operatorname{Im} G_k(+0 + i\alpha) d\alpha, : k = 1, 2$$

are finite on $[0, \varepsilon]$. The measure produced by

$$G_4 = \frac{(2i\lambda)^2 m_+}{m_- + m_+} \omega = \frac{2i\lambda}{m_- + m_+} \cdot 2i\lambda m_+ \omega$$

requires a bit more care. Since the factor $\frac{2i\lambda}{m_- + m_+}$ has already been analyzed above, one needs to make sure that $\alpha m_+ (-\alpha^2) \omega(i\alpha)$ stays bounded as $\alpha \rightarrow 0$. From (8.6)

$$-\alpha m_+ (-\alpha^2) = -\alpha^2 + \frac{2\alpha^2}{1 + L_+(i\alpha)}. \quad (8.7)$$

It follows from (3.11) and (3.9) that

$$\begin{aligned} 1 + L_+(i\alpha) &= 1 - \frac{\int_0^\infty e^{-2\alpha x} V(x) y_+(x, i\alpha) dx}{2\alpha + \int_0^\infty V(x) y_+(x, i\alpha) dx} \\ &= \frac{2\alpha + \int_0^\infty V(x) (1 - e^{-2\alpha x}) y_+(x, i\alpha) dx}{2\alpha + \int_0^\infty V(x) y_+(x, i\alpha) dx} \\ &= 2\alpha \frac{1 + \int_0^\infty V(x) \frac{1 - e^{-2\alpha x}}{2\alpha} y_+(x, i\alpha) dx}{2\alpha + \int_0^\infty V(x) y_+(x, i\alpha) dx} \end{aligned}$$

where $y_+(x, i\alpha)$ solves the integral equation (3.8)

$$y_+(x, i\alpha) = 1 + \int_x^\infty \frac{1 - e^{-2\alpha(s-x)}}{2\alpha} V(s) y_+(s, i\alpha) ds. \quad (8.8)$$

Hence for the second term on the right hand side of (8.7) we have

$$\begin{aligned} \frac{2\alpha^2}{1 + L_+(i\alpha)} &= \frac{\alpha \left(2\alpha + \int_0^\infty V(x) y_+(x, i\alpha) dx \right)}{1 + \int_0^\infty \frac{1 - e^{-2\alpha x}}{2\alpha} V(x) y_+(x, i\alpha) dx} \\ &\leq \frac{\alpha \left(2\alpha + \int_0^\infty |V(x)| |y_+(x, i\alpha)| dx \right)}{\left| 1 - \int_0^\infty \frac{1 - e^{-2\alpha x}}{2\alpha} |V(x)| |y_+(x, i\alpha)| dx \right|}. \end{aligned} \quad (8.9)$$

From (8.8) one has

$$\begin{aligned} |y_+(x, i\alpha)| &\leq 1 + \int_x^\infty \frac{1 - e^{-2\alpha(s-x)}}{2\alpha} |V(s)| |y_+(s, i\alpha)| ds \\ &\leq 1 + \int_x^\infty (s-x) |V(s)| |y_+(s, i\alpha)| ds \\ &\leq 1 + \int_x^\infty s |V(s)| |y_+(s, i\alpha)| ds. \end{aligned}$$

Iterating this inequality immediately produces

$$\begin{aligned} |y_+(x, i\alpha)| &\leq \sum_{n \geq 0} \left(\int_0^\infty x |V(x)| dx \right)^n \\ &\leq \sum_{n \geq 0} \|V\|_{L_+^1(\langle x \rangle dx)}^n = \frac{1}{1 - \|V\|_{L_+^1(\langle x \rangle dx)}}. \end{aligned}$$

Denoting $\varepsilon = \|V\|_{L_+^1(\langle x \rangle dx)}$ and taking it small enough, the inequality (8.9) then yields

$$\begin{aligned} \frac{2\alpha^2}{1 + L_+(i\alpha)} &\leq \frac{\alpha}{1 - \frac{1}{1-\varepsilon} \int_0^\infty x |V(x)| dx} \left(2\alpha + \frac{1}{1-\varepsilon} \int_0^\infty |V(x)| dx \right) \\ &\leq \frac{2\alpha + \varepsilon}{1 - 2\varepsilon} \alpha \rightarrow 0, : \alpha \rightarrow 0. \end{aligned}$$

Thus the measure $d\rho$ is finite.

We now prove the Properties (1)-(4). We start by splitting

$$\mathbb{M}_{z,t} = \mathbb{M}_{z,t}^+ + \mathbb{G}_{z,t} \quad (8.10)$$

where $\mathbb{M}_{z,t}^+$ is the Marchenko type operator introduced in Definition 7.1 with

$$\begin{aligned} d\rho(\alpha) &= e^{-2\alpha z + 8\alpha^3 t} \sum_{n=1}^N (c_n^+)^2 \delta(\alpha - \kappa_n^+) d\alpha, \\ \phi(\lambda) &= e^{2i\lambda x + 8i\lambda^3 t} R_+(\lambda), \end{aligned}$$

and $\mathbb{G}_{z,t}$ defined by (7.4) – (7.6) with $G = \frac{R_-}{1 - R_- L_+} T_+^2$. Let us show that G satisfies all the conditions of Proposition 7.2. Indeed each function R_- , L_+ , T_+ is clearly subject to conditions (i)-(iv) of Proposition 7.2. The function G is then immediately subject to conditions (i)-(ii) of Proposition 7.2. The existence of boundary values of G on the real line is also obvious. It is also in L^∞ since

$$|G(\lambda + i0)| = |R(\lambda) - R_+(\lambda)| \leq 2.$$

Thus G satisfies condition (iii) of Proposition 7.2. Condition (iv) was verified above. By Proposition 7.2 (2) $\mathbb{G}_{z,t}$ is then selfadjoint for real z 's for any $t > 0$. As was shown above $\mathbb{M}_{z,t}^+$ is Marchenko operator corresponding to V_+ and its selfadjointness for real z 's is a well-known fact. Thus $\mathbb{M}_{z,t}$ is selfadjoint for real z and positive t . The boundedness of $\mathbb{M}_{z,t}$ follows from the finiteness of $d\rho$ (see, e.g. [43]). This proves Property 1. Turn now to Property 2. Under condition $V_+ \in L_+^2(e^{\delta x^{1/2}} dx)$ for some $\delta > 0$, the operator $\mathbb{M}_{z,t}^+$ is real analytic in z for any

$t > 0$ [51] and by Proposition 7.2 (4) $G_{z,t}$ is entire. Therefore, $M_{z,t}$ is real analytic and [49] $(I + M_{z,t})^{-1}$ is real meromorphic in z for any $t > 0$. By Lemma 6.2, $G_{z,t} \in \mathfrak{S}_1$ and by Lemma 6.3, $M_{z,t}^+$ is a trace type operator. Thus Property 2 is proven. Note that if $V_+ = 0$ then $M_{z,t}^+ = 0$ and Property 3 immediately follows.

Assume now that $-\partial_x^2 + V$ has a nontrivial a.c. component S of multiplicity 2 and hence, by Lemma 5.4, $|R(\lambda)| < 1$ a.e. on S . By Lemma 7.3, $I + M_{z,t}$ is boundedly invertible for any real z and $t \geq 0$ and Property 4 is proven. \square

Remark 8.2. If $V(x) = -h^2\chi_-(x)$ then

$$R(\lambda) = - \left(\frac{h}{\lambda + \sqrt{\lambda^2 + h^2}} \right)^2$$

and ρ is absolutely continuous, supported on $[0, h]$ and

$$d\rho(\alpha) = \frac{2}{\pi h^2} \alpha \sqrt{h^2 - \alpha^2} d\alpha.$$

Remark 8.3. Marchenko operators for the cases of V 's such that $V_-(x) - p(x) \in L_-^1(\langle x \rangle dx)$ with either $p(x) = \text{const}$ or periodic and $V_+(x) \in L_+^1(\langle x \rangle dx)$ (so-called step-like potentials) have been considered by many authors in the connection with inverse problems (see, e.g., [4]) and IST for KdV ([25], [7], [53], [35], [14], etc.), scattering quantities being typically introduced differently from ours.

Remark 8.4. If $V_+ \in L_+^1(e^{\delta x} dx)$ for some $\delta > 0$ then $M_{z,t}$ is entire and $(I + M_{z,t})^{-1}$ is meromorphic in z on \mathbb{C} for any $t > 0$.

Remark 8.5. Loosely speaking, the measure ρ carries over the information about the negative spectrum and R does it for the positive spectrum but they need not be independent.

8.2. Determinant solution to the Cauchy problem for the KdV equation. Our main result is given in the following theorem.

Theorem 8.6. Let real V_0 be such that $\text{Spec}(-\partial_x^2 + V_0)$ is bounded from below and has a non-empty twofold a.c. spectrum. Assume that

$$\chi_- V_0 \in L_-^2(e^{-\delta_- |x|} dx) \quad (8.11)$$

$$\chi_+ V_0 \in L_+^2(e^{\delta_+ x^{1/2}} dx) \quad (8.12)$$

for some $\delta_{\pm} > 0$. Then the Cauchy problem for the KdV equation

$$\begin{cases} \partial_t V - 6V\partial_x V + \partial_x^3 V = 0 \\ V(x, 0) = V_0(x) \end{cases} \quad (8.13)$$

has a unique global natural solution $V(x, t)$ given by

$$V(x, t) = -2\partial_x^2 \log \text{Det}(I + M_{x,t}) \quad (8.14)$$

where $M_{x,t}$ is defined in Theorem 8.1, $V(x, t)$ being a real analytic function in x for any $t > 0$. Moreover, for any $a > -\infty$

$$\lim_{t \rightarrow +0} \|V(\cdot, t) - V_0\|_{L^2(a, \infty)} = 0. \quad (8.15)$$

Proof. Let $V_{0,n}(x)$ be an arbitrary sequence of real compactly supported L^2 functions such $\text{Supp } V_{0,n} = (a_n, b_n)$ and

$$\|V_{0,n} - V_0\|_{L^2_{\text{loc}}} \rightarrow 0, : n \rightarrow \infty.$$

Then the KdV equation with initial data $V_{0,n}(x)$ has a unique solution $V_n(x, t)$ computed by the standard inverse scattering transform

$$V_n(x, t) = -2\partial_x^2 \log \text{Det}(I + \mathbb{M}_{n,x,t}),$$

where $\mathbb{M}_{n,x,t}$ is the Marchenko operator corresponding to $V_{0,n}$. By Theorem 8.1, each $V_n(x, t)$ is a meromorphic function in x on the entire complex plane. Consider the function $V(x, t)$ given by (8.14). By Theorem 8.1, it is well defined and real analytic in x for any $t > 0$ and it remains to prove that it solves (8.13). To this end we rewrite $V = V_n + \Delta V_n$, where $\Delta V_n := V - V_n$, and insert this into the left hand side of (8.13):

$$\partial_t V - 6V\partial_x V + \partial_x^3 V = \partial_t \Delta V_n + 3\partial_x [(\Delta V_n - 2V)\Delta V_n] + \partial_x^3 \Delta V_n.$$

Using (8.10) by a straightforward computation we have (dropping subscripts x, t)

$$\begin{aligned} \Delta V_n(x, t) &= \Delta V_n^+(x, t) \\ &\quad + 2\partial_x^2 \log \text{Det} \left(I + (I + \mathbb{M}_n^+)^{-1} \mathbb{G}_n \right) \left(I + (I + \mathbb{M}^+)^{-1} \mathbb{G} \right)^{-1} \\ &= \Delta V_n^+(x, t) \\ &\quad + 2\partial_x^2 \log \det \left\{ I - (I + \mathbb{M})^{-1} (I + \mathbb{M}^+) (I + \mathbb{M}_n^+)^{-1} \Delta \Omega \right\} \end{aligned} \quad (8.16)$$

with

$$\Delta \Omega = \Delta \mathbb{G}_n - \Delta \mathbb{M}_n^+ (I + \mathbb{M}^+)^{-1} \mathbb{G},$$

where the determinant on the right hand side of (8.16) is understood in the usual way because by Lemma 6.2 both $\Delta \mathbb{G}$ and \mathbb{G} are trace class, and V_n^+ stands for the solution to the KdV equation with the initial profile $\chi_+ V_{0,n}$. Since $V_n^+(x, t) \rightarrow V^+(x, t)$ uniformly in x as $n \rightarrow \infty$ (one of the main results of [51]), the first term on the right hand side of (8.16) vanishes as $n \rightarrow \infty$. We now need to show that so does the other one. But by [51] $\|\Delta \mathbb{M}_n^+\| \rightarrow 0, n \rightarrow \infty$, and therefore

$$\left\| \Delta \mathbb{M}_n^+ (I + \mathbb{M}^+)^{-1} \mathbb{G} \right\|_{\mathfrak{S}_1} \leq \left\| (I + \mathbb{M}^+)^{-1} \right\| \left\| \mathbb{G} \right\|_{\mathfrak{S}_1} \left\| \Delta \mathbb{M}_n^+ \right\| \rightarrow 0, : n \rightarrow \infty, \quad (8.17)$$

$$\begin{aligned} \left\| (I + \mathbb{M}_n^+)^{-1} \right\| &\leq \left\| (I + \mathbb{M}^+)^{-1} \right\| \left\| \left(I - (I + \mathbb{M}^+)^{-1} \Delta \mathbb{M}_n^+ \right)^{-1} \right\| \\ &\leq \left\| (I + \mathbb{M}^+)^{-1} \right\| \left(1 - \left\| (I + \mathbb{M}^+)^{-1} \right\| \left\| \Delta \mathbb{M}_n^+ \right\| \right)^{-1} \\ &\rightarrow \left\| (I + \mathbb{M}^+)^{-1} \right\|, : n \rightarrow \infty. \end{aligned} \quad (8.18)$$

Next, by Lemma 6.2 (reinstating subscripts x, t)

$$\begin{aligned} \|\Delta \mathbb{G}_{n,x,t}\|_{\mathfrak{S}_1} &\lesssim \left\| \frac{\Delta g_{n,x,t}(\lambda)}{\text{Im } \lambda} \right\|_{L^1(\Gamma)} = \left\| \frac{e^{2i\lambda x} e^{8i\lambda^3 t} \Delta G_n(\lambda)}{\text{Im } \lambda} \right\|_{L^1(\Gamma)} \\ &\leq \left\| \frac{e^{2i\lambda x} e^{8i\lambda^3 t} \Delta G_n(\lambda)}{\text{Im } \lambda} \right\|_{L^1(\Gamma_N)} + \left\| \frac{e^{2i\lambda x} e^{8i\lambda^3 t} \Delta G_n(\lambda)}{\text{Im } \lambda} \right\|_{L^1(\Gamma \setminus \Gamma_N)} \end{aligned} \quad (8.19)$$

where Γ is as in Figure 1 and $\Gamma_N := \Gamma \cap \{\lambda : |\lambda| \leq N\}$. We need to show that each term on the right hand side of (8.19) is small for large n and N . For ΔG_n we have

$$\begin{aligned} \Delta G_n &= \Delta \frac{R_{n,-}}{1 - R_{n,-}L_{n,+}} T_{n,+}^2 \\ &= \Delta \left(\frac{T_{n,+}^2}{L_{n,+}} \right) \left(\frac{1}{1 - R_{n,-}L_{n,+}} - 1 \right) + \frac{T_{n,+}^2}{L_{n,+}} \frac{L_{n,+}\Delta R_{n,-} + R_{n,-}\Delta L_{n,+}}{(1 - R_{n,-}L_{n,+})(1 - R_{n,-}L_{n,+})}. \end{aligned}$$

But [8]

$$\|V_{0,n} - V_0\|_{L_{\text{loc}}^2} \rightarrow 0 \implies m_{\pm,n,0}(z) \rightarrow m_{\pm,0}(z), \quad n \rightarrow \infty,$$

the latter convergence being uniform in z on compacts in \mathbb{C}_+ . Due to (5.2) and (5.3) then $\Delta R_{n,-} \rightarrow 0$, $\Delta L_{n,+} \rightarrow 0$, $n \rightarrow \infty$, also uniformly on compacts in \mathbb{C}_+ . Since by [51] we also have $\Delta T_{n,+} \rightarrow 0$, $n \rightarrow \infty$, we conclude that $\Delta G_n \rightarrow 0$, $n \rightarrow \infty$ uniformly on Γ_N and hence the first norm on the right hand side of (8.19) is small for n large enough. The second norm on the right hand side of (8.19) is small if N is large enough due to the decay of $e^{8i\lambda^3 t}$ on Γ and one concludes from (8.19) that

$$\|\Delta G_{n,x,t}\|_{\mathfrak{S}_1} \rightarrow 0, \quad n \rightarrow \infty \quad (8.20)$$

for any real (and complex too) x and $t > 0$. Combining now (8.17), (8.18), (8.20) and taking into account [48]

$$|\det(I + A) - \det(I + B)| \lesssim \|A - B\|_{\mathfrak{S}_1} e^{(1+\|A\|_{\mathfrak{S}_1}+\|B\|_{\mathfrak{S}_1})}$$

we conclude that

$$\det \left\{ I - (I + \mathbb{M}_{x,t})^{-1} \left(I + \mathbb{M}_{x,t}^+ \right) \left(I + \mathbb{M}_{n,x,t}^+ \right)^{-1} \Delta \Omega_{x,t} \right\} \rightarrow 0, \quad n \rightarrow \infty$$

uniformly in x on any compact interval for any $t > 0$. Note that if a sequence of real analytic functions $f_n(z)$ converges to some real analytic function $f(z)$ uniformly on a compact set K then $f'_n(z) \rightarrow f'(z)$ also uniformly on K . Indeed, it immediately follows from the Cauchy formula that for any function f analytic on some closed disc $B_r(a) = \{z : |z - a| \leq r\}$ one has

$$\|f'\|_{L^\infty(B_r(a))} \lesssim \frac{\|f\|_{L^\infty(B_r(a))}}{(1-\varepsilon)r}, \quad 0 < \varepsilon < 1.$$

One can now easily conclude that the second term on the right hand side of (8.16) converges to zero uniformly on compacts in \mathbb{R} as $n \rightarrow \infty$ and the function $V(x, t)$ formally defined by (8.14) is indeed a classical solution to the KdV problem. It remains to demonstrate (8.15). To this end one needs to use the Fredholm expansion of the determinant in (8.14). The complete expansion is unwieldy but [46] its first term is the least regular. Moreover, its component corresponding to $\chi_+ V_0$ is classical and the problem essentially boils down to showing that the L_+^2 norm of

$\int_{\Gamma} i\lambda e^{2i\lambda x} (e^{8i\lambda^3 t} - 1) G(\lambda) d\lambda$ vanishes as $t \rightarrow 0$ in L_+^2 . We have

$$\begin{aligned} & \left\| \int_{\Gamma} i\lambda e^{2i\lambda x} (e^{8i\lambda^3 t} - 1) G(\lambda) d\lambda \right\|_{L_+^2} \\ & \leq \int_{\Gamma} |\lambda| \left\| e^{2i\lambda x} \right\|_{L_+^2(dx)} |e^{8i\lambda^3 t} - 1| |G(\lambda)| |d\lambda| \\ & = \int_{\Gamma} \frac{|\lambda|}{2\sqrt{\operatorname{Im} \lambda}} |e^{8i\lambda^3 t} - 1| |G(\lambda)| |d\lambda| \\ & \lesssim \int_{\Gamma} \sqrt{\operatorname{Im} \lambda} |e^{8i\lambda^3 t} - 1| |G(\lambda)| |d\lambda| \\ & \lesssim \int_{\Gamma} \sqrt{\operatorname{Im} \lambda} |e^{8i\lambda^3 t} - 1| |R_-(\lambda)| |d\lambda|. \end{aligned}$$

To apply the Lebesgue dominated convergence theorem one merely needs to verify that $(\operatorname{Im} \lambda)^{-1/2} R_-(\lambda) \in L^1(\Gamma)$ or, due to (5.2), that $(\operatorname{Im} \lambda)^{-1/2} (i\lambda - m_-(\lambda^2)) \in L^1(\Gamma)$. But [8]

$$|i\lambda - m_-(\lambda^2)| \lesssim \left| \int_{\mathbb{R}_-} e^{2i\lambda|x|} V_{0,-}(x) dx \right| + \frac{1}{|\lambda|^2} \left| \int_{\mathbb{R}_-} e^{-2\operatorname{Im} \lambda|x|} V_{0,-}(x)^2 dx \right|$$

and hence, due to (8.11),

$$(\operatorname{Im} \lambda)^{-1/2} \int_0^\infty e^{2i\lambda x} V_{0,-}(-x) dx \in L^1(\Gamma) \quad (8.21)$$

is to be demonstrated. But, in virtue of condition (8.11)

$$\begin{aligned} \int_1^\infty e^{-\operatorname{Im} \lambda x} |V_{0,-}(-x)| dx &= e^{-\operatorname{Im} \lambda} \int_0^\infty e^{-\operatorname{Im} \lambda x} |V_{0,-}(-x-1)| dx \\ &= O(e^{-\operatorname{Im} \lambda}), \quad \operatorname{Im} \lambda \rightarrow \infty, \end{aligned} \quad (8.22)$$

and [45]

$$\int_0^1 e^{-2\operatorname{Im} \lambda x} V_{0,-}(-x) dx = \frac{V_{0,-}(-0)}{2\operatorname{Im} \lambda} + o\left(\frac{1}{\operatorname{Im} \lambda}\right), \quad \operatorname{Im} \lambda \rightarrow \infty, \quad (8.23)$$

provided that $x = 0$ is a Lebesgue point of $V_0(x)$. Since $V_0(x)$ is locally integrable, almost every x is a point of Lebesgue continuity and thus, without loss of generality, we may assume that $a = 0$ is Lebesgue. Putting (8.23) and (8.22) together implies (8.21) and thus (8.15) is proven for $a = 0$. Simple shifting arguments extend (8.15) to any real a . \square

Theorem 8.6 extends our main result in [47] where somewhat weaker solutions are considered under much stronger conditions on initial data. In addition no explicit formula (8.14) is given there either. Theorem 8.6 considerably extends also [51] where similar results (save (8.14)) are obtained. In [51] $\chi_- V_0$ is assumed to be Faddeev and from L_-^2 which is of course much stronger than (8.11). The approach of [51] is also based upon inverse scattering but the analysis is conducted within the classical Marchenko theory. We however crucially used [51] to combine the treatment of the \mathbb{R}_- from [47] with the one given in [51] for \mathbb{R}_+ .

Remark 8.7. Note that Conditions (8.11) and (8.12) alone accept those V_0 's for which $|V_0(x)| \rightarrow \infty$ exponentially fast as $x \rightarrow -\infty$ but exhibit a subexponential decay as $x \rightarrow$

∞ . Condition $\inf \text{Spec}(-\partial_x^2 + V_0) \geq -h_0 > -\infty$, meaning that $V_0(x)$ is essentially bounded from below, however does not allow $V_0(x) \rightarrow -\infty$ as $x \rightarrow -\infty$. The condition that $\text{Spec}_{ac}(-\partial_x^2 + V_0)$ has a non-empty component of multiplicity two, on the other hand, does allow $V_0(x)$ to go to $-\infty$ as $x \rightarrow -\infty$ but not to ∞ . This condition assumes certain pattern of behavior of V_0 when $x \rightarrow -\infty$ but it is hard to express in terms of V_0 alone. Loosely speaking, for an operator with a.c. spectrum, it is possible to approximately predict future values of the potential, with arbitrarily high accuracy, based on information about past values (so-called Oracle Theorems). We refer the reader to [44] and the literature cited therein. On the other hand, there are many explicit examples of potentials (including those decaying like $|x|^{-\alpha}$, $\alpha < 1/2$) for which the spectrum is purely singular (see, e.g. [41], [36]).

Remark 8.8. If we assume that V_0 is short range then the right reflection coefficient is also well-defined by (3.10). The formula (3.10) says that $R_+(\lambda)$ admits an analytic continuation into \mathbb{C}_- but does not imply analyticity in \mathbb{C}_+ .

Remark 8.9. Theorem 8.6 holds with an obvious change in (8.14) if we replace 8.12) with $\chi_+ V_0 - c \in L_+^2(e^{\delta_+ x^{1/2}} dx)$ with some real c . Indeed, performing a simple Galilean transform, one has

$$V(x, t) = c + W(x, t)$$

where W solve the KdV equation with initial data W_0 subject to the conditions of Theorem 8.6.

Remark 8.10. The decay condition (8.12) in Theorem 8.6 can likely be relaxed to read $\chi_+ V_0 \in L^1(\langle x \rangle dx)$ (or at least to $L^1(\langle x \rangle^2 dx)$) but $V(x, t)$ will no longer be a real analytic function in x for any $t > 0$. Further relaxation of the decay condition at $+\infty$ runs into serious problems which will be discussed in the next section.

9. DISCUSSIONS, COROLLARIES, AND OPEN PROBLEMS

9.1. Hirota τ -function. The explicit formula (8.14) is by no means new. In the context of integrable systems the formula

$$V(x, t) = -2\partial_x^2 \log f(x, t) \tag{9.1}$$

seems to have appeared first in the physical literature [24] back in early 70s and has become one of the main ingredient of soliton theory. (or perhaps even earlier in the connection with N soliton solutions.) The nature of $f(x, t)$ varies depending on the context where it appears in. It is used, e.g., as a substitution to transform the KdV equation into the so-called bilinear KdV equation (Hirota's τ -transform). Every known explicit solution can be written in the form (9.1) where $f(x, t)$ is a certain Wronskian (see, e.g. [37]) or finite determinant [3]. It is also well known in certain contexts similar to ours (see, e.g. [12], [53]) and is referred to as determinant, Bargman, Dyson, etc. We however call it Hirota's τ -representation. The literature on such formulas is enormously broad but in our generality (8.14) appears to be new. We however were unable to prove that $\mathbb{M}_{x,t}$ is actually trace class for any $t > 0$ and the determinant in (8.14) can therefore be understood in the classical Fredholm sense. We could not find a rigorous proof of this fact in the literature even for short range initial profiles. The problem is equivalent to the

question what conditions on (short range) V_0 guarantee that the integral Hankel operator \mathbb{H} on L^2_+ defined by

$$(\mathbb{H}f)(x) = \int_{\mathbb{R}_+} H(x+y) f(y) dy, \quad x \in \mathbb{R}_+.$$

were

$$H(x) := \int e^{i\lambda x} e^{i\lambda z + i\lambda^3 t} R(\lambda) d\lambda, \quad (9.2)$$

is a trace class operator for any real z and $t > 0$. Peller found [43] a complete characterization of all trace class Hankel operators in terms of a very subtle smoothness of H (membership of its Fourier transform in the so-called analytic Besov class). However this characterization cannot be easily expressed in term of V_0 .

9.2. Asymptotic solutions. The formula (8.14) (and Theorem 8.6 as a whole) is particularly convenient to study solutions of Cauchy problems for the KdV equation. It has however been primarily used in the context of short range reflectionless initial profiles where the Marchenko operator becomes finite rank and the Fredholm determinant turns into a linear algebra object. Historically, this case was studied first and the phenomenon of nonlinear superposition and interaction of solitons was understood this way. This analysis was then extended in [18] to the closure of reflectionless potentials. In the form closest to ours (8.14) seems to have appeared in [53]. Namely, under the assumption that $V_0 \rightarrow -c^2$ as $x \rightarrow -\infty$ and $V_0 \rightarrow 0$ as $x \rightarrow \infty$ sufficiently fast it was used in [53] to derive and analyze a long time asymptotic of $V(x, t)$ producing short-cuts to most of previous results of [25]. The treatment was justified under an extra assumption that $R = 0$ which however holds only asymptotically as $t \rightarrow \infty$. In [35] this condition was not imposed and a rigorous justification was done assuming only that $\partial_x V_0$ is from the Schwartz class. We emphasize that these works are concerned with asymptotic solutions and WP issues are not treated there. This is, in fact, the main focus of the present paper. We however believe that Theorem 8.6 should be particularly useful to derive various asymptotics generalizing and improving previously known results¹⁵ in the conditions it is proven under. It could be achieved by a suitable approximation of the operator $\mathbb{M}_{x,t}$ in (8.14) which can be done in a number of different ways. We hope to return to this important issue elsewhere.

9.3. Conditions on a.c. spectrum. The condition that $\text{Spec}(-\partial_x^2 + V_0)$ is bounded from below guarantees that no blow-up solitons will instantaneously emerge from $-\infty$. It is however may be possible to construct an initial condition V_0 such that the negative spectrum of $-\partial_x^2 + V_0$ is very sparse but not bounded from below so that Theorem 8.6 would still hold. The physical content of such a situation would be dubious though.

The condition that $\text{Spec}(-\partial_x^2 + V_0)$ has a non-trivial a.c. component of multiplicity two does not appear to be physically motivated. While satisfied in all previously studied cases mentioned in Remark 8.3 this condition is hard to verify in general and only vague statements can be made as of today explaining what it means for V_0 to satisfy such condition. Theorem 8.1 guarantees that the determinant in (8.14) does not vanish on the real line and hence $V(x, t)$ has no real poles. The latter means no blow-up solutions develop over finite time. However

¹⁵Roughly speaking, a typical such result says that $V(x, t)$ asymptotically splits as $t \rightarrow \infty$ into an infinite train of solitons of height $-2c^2$.

as mentioned above one can explicitly construct initial profiles (e.g. Pearson blocks running to $-\infty$ and zero on the right) for which the spectrum is positive and its a.c. component fills \mathbb{R}_+ but has uniform multiplicity one. Lemma 7.3 does not apply as $|R(\lambda)| = 1$ a.e. on the real line. Note that this problem has embedded dense singular spectrum. Yet another example comes from a white noise type random initial profile supported on the left half line. We however conjecture that the assertion of Lemma 7.3 would hold. In a somewhat simplified case (absence of negative spectrum), the problem essentially boils down to the following question. If the operator

$$I + \chi_+ \mathcal{F} \phi \mathcal{F} : L_+^2 \rightarrow L_+^2$$

is boundedly invertible for any unimodular function ϕ of the form

$$\phi(\lambda) = e^{i(\lambda^3 t + c\lambda)} I(\lambda)$$

where $t \geq 0, c \in \mathbb{R}$ are constants and $I(\lambda)$ is an inner function of the upper half plane (i.e. $I \in H_+^\infty$ and such that $|I(\lambda)| < 1$ in \mathbb{C}_+ and $|I(\lambda)| = 1$ a.e. on the real line)? This problem can be viewed as a Riemann-Hilbert problem or a problem about invertibility of the Toeplitz operator with the (unimodular) symbol ϕ . Since $e^{i\lambda^3 t}$ is not a Nevanlinna function, the powerful machinery of Riemann-Hilbert problem or Hankel/Toeplitz operators does not readily apply. Moreover, the Bourgain factorization theorem for a unimodular function [6] suggests that the answer may actually be negative¹⁶ meaning that, loosely speaking, positon solutions (with double pole singularities) will emerge from noise. The latter would mean that either shallow water rogue waves are likely a stochastic phenomenon or the KdV equation does not model the situation good enough in this case. We however cautiously conjecture that the answer is affirmative and moreover the conclusion of Theorem 8.6 holds without the hard-to-verify condition on the a.c. spectrum. An improved theorem 8.6 would give solutions with simple a.c. and dense singular spectra (produced, e.g., by Pearson blocks) which would of course be a new type of KdV solutions.

9.4. Uniqueness results. The following statement is a direct consequence of the analyticity of $V(x, t)$ for $t > 0$.

Corollary 9.1. *Under conditions of Theorem 8.6 the solution $V(x, t)$ can not vanish on a set of positive Lebesgue measure for any $t > 0$ unless V_0 is identically zero.*

This extends a result due to Zhang [55] stating that if $V(x, t)$ is a solution of the KdV equation then it cannot have compact support at two different moments unless it vanishes identically. Corollary 9.1 quickly recovers and improves yet another important result of [55] saying that, assuming $V_0 \in L^1(\langle x \rangle dx)$ and $\partial_x V_0 \in L^1$, once $\text{Supp } V_0(x, t_0), \text{Supp } V(x, t_1) \subset (-\infty, a)$ for $t_0 < t_1$ then $V(x, t) = 0$. The techniques of [55] use the Marchenko theory and some Hardy space arguments. Zhang's approach has been simplified and generalized in recent [15]. The main results of [55] were extended in [13]. Under certain regularity and decay conditions, the main result of [13] says that if at two distinct times two solutions differ by super exponentially decaying functions then the solutions must coincide. We believe that this result can be improved by our techniques and will return to this

¹⁶We thank Donald Marshall for bringing out attention to this paper.

elsewhere. Note that [13] treats a more general class of KdV type equations for which the IST need not apply.

9.5. Analyticity. It is proven in [11] that if $V_0(x)$ is analytic in the strip $|\operatorname{Im} z| < a$ and has Schwartz decay there, then $V(x, t)$ is meromorphic in a strip with at most N poles (where N is the number of bound states of $-\partial_x^2 + V_0(x)$) off the real line. Note that in this situation the reflection coefficient R necessarily exhibits an exponential decay on the real line. This of course need not occur in our case (even if we assume that V_0 is supported on \mathbb{R}_-) and our real meromorphic solution has, in general, infinitely many poles for any $t > 0$ in any strip around the real line (but not on it) accumulating only to infinity. These poles analytically depend on t and hence may appear or disappear only on the boundary of analyticity of $V(x, t)$. In the case of V_0 supported on \mathbb{R}_- , $V(x, t)$ is meromorphic on the whole complex plane meaning that the poles can only move. In fact, the importance of pole dynamics was recognized by Kruskal [30] back in the early 70s for pure soliton solutions and has been actively studied since then (see also [2], [10], [22], [5] to mention just four). Now existence of meromorphic solutions is referred to as the Painlevé property (see, e.g. [39] and [26]) Theorem 8.6 says the KdV with the initial profile V_0 supported on \mathbb{R}_- has the Painlevé property. However we cannot say much about the structure of our poles and their dynamics and we don't know how one can spot the future poles from the (non-analytic) initial profile and how they actually appear. In particular, we have no norm estimates or any other quantitative way to express the 'size' for this function in terms of initial data. Such estimates could likely follow from conservation laws which typically accompany the IST. For initial data which tends to a constant at $-\infty$ deriving such formulas should not be a problem. We are not sure if conservation laws have been derived for general step like initial data. We do not know if our real meromorphic solution is meromorphic on the whole complex plain. The answer depends the same question in the setting of [51]. The techniques used in [51] yield only local analyticity around the real line (real analyticity) and cannot be easily adjusted to obtain global analyticity. We however believe that it is achievable but by different methods. In [39] it is claimed without a proof that $V(x, t)$ is meromorphic for any $t > 0$ under the only assumption that $V_0 \in L^1(\langle x \rangle^2 dx)$. This would likely be the case if the Fourier transform of V_0 admitted an analytic continuation into the lower half plane. Unfortunately, an arbitrary function from $L^1(\langle x \rangle^2 dx)$ doesn't have this property¹⁷.

We have not looked into the question if there are nontrivial initial profiles for which $V(x, t)$ is an entire function of x for any $t > 0$. The answer is likely affirmative as the linear part of the KdV equation (Airy's equation) has a very strong smoothing property. It is not hard to show that this equation instantaneously evolves a multisoliton initial profile, which is a meromorphic functions on the whole complex with infinitely many double poles, into an entire function. On the other hand the non-linear part of the KdV (Hopf's equation) tends to break analyticity.

¹⁷We owe Eugene Korotyeve for this comment.

9.6. Slower or no decay at $+\infty$. Due to the directional anisotropy of the KdV equation (related to the different decay behavior of the Airy functions) we cannot switch in Theorem 8.6 $V_0(x)$ to $V_0(-x)$. Theorem 8.6 says that initial data on $(-\infty, a)$ instantaneously evolves into a meromorphic function (the so-called gain of regularity) for any finite a and causes no problem to WP. Contrary to this, even small data on (a, ∞) may reduce smoothness of a solution (loss of regularity) and even lead to a blow-up solution. This phenomenon can be drastically seen in the case of a delta initial profile which evolves into a single soliton plus an Airy type function. Due to the invariance of the KdV equation with respect to $(x, t) \rightarrow (-x, -t)$, one has an example of a smooth (meromorphic) initial profile rapidly decaying at $-\infty$ and exhibiting slow oscillatory decay at $+\infty$ which turns at a certain instant of time into a delta function (focusing effect). This delta function (which could formally be considered as a compactly supported initial data) then instantaneously evolves into a meromorphic function which stays meromorphic for all times. Note that since the delta function is in $H^{-3/4}$ the general theory [50] guarantees that this problem is globally WP. This makes one believe that IST techniques should work in the general setting of $H^{-3/4}$ data¹⁸. However, no rigorous IST method, to the best of our knowledge, is developed in this case in full generality. Comparing to the classical IST, there are new circumstances like infinite negative spectrum and a rich singular embedded positive spectrum in this setting. The importance of this problem was recognized back in the 80s and the first essential progress was made in [34] where the IST was extended to certain Wigner-von Neumann initial profiles producing finitely many simple real zeros of the transmission coefficient. This topic has been continued in numerous publications (see, e.g. [38], [31] and the literature cited therein). The main issue here is the appearance of the so-called positon solutions. These solutions are in H^{-2} but exhibit Wigner-von Neumann decay at infinity. The Schrodinger operator with a H^{-2} potential can be defined in a few nonequivalent ways resulting in, at least, two nonequivalent IST formalisms (see [32] and the relevant discussions therein).

The situation is much better in the context of initial data with periodic type behavior at $+\infty$ (see recent [14] and a very detailed account of the literature therein). As discussed above, suitable analogs of the IST are available in this setting which allow to push WP results to the periodic H^{-1} space. However the KdV flow preserves the smoothness of a periodic initial profile confirming that the data on (a, ∞) works against the gain of regularity effect¹⁹.

We however cautiously conjecture that the Hirota τ -representation (8.14) holds in one form or another for any well-posed KdV problem. This would mean that KdV equation is completely (and explicitly) integrable for a large scope of problems. The arguments used in the proof of Theorem 8.6 no longer apply to profiles with no decay at $+\infty$ ²⁰. It would be extremely interesting to understand this situation even in the case of the data supported on $(0, \infty)$.

¹⁸Better yet, the recent paper [28] says that the KdV equation is WP for a certain subset of H^{-1} functions. More specifically, the image of L^2 under the Miura transform.

¹⁹We don't know how the interference between the data $(-\infty, a)$ and (a, ∞) affects regularity either.

²⁰Due to Remark 8.9, initial data approaching a constant at $+\infty$ is not of interest to us.

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